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OF THE

FIRST DISTRICT NORMAL SCHOOL

KIRKSVILLE, MISSOURI

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Mathematics Society, 1915

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FOREWORD

To the Teachers of Missouri:

The Mathematics Department of the First District Normal School prepares this Bulletin in the hope that it may present and discuss some topics that are both interesting and suggestive, that it will raise and answer some of the questions that confront the teachers of Mathematics in Northeast Missouri, and that to all readers it may be a source of inspiration for mathematical thought and investigation.

THE USE OF THE GRAPH IN GEOGRAPHY TEACHING

By Byron Cosby

Cross section or graph paper serves a very practical purpose in mathematics in making definite statements. It can be used very satisfactorily in geography work in the map drawing. For the more elementary grades the unit scale may be the city block, town lot or the part of a mile. The base line should be ruled heavy or in a different color than the location lines. The base lines should cross at the location of the school house, or at some well known point in the neighborhood of the house. With the school house as a starting point well known places may be located, as the town hall, the churches, railroad stations, and homes of the members of the class. If the school is in the country, the nearby villages and farm houses may be located. Or if the grade is more advanced, the trees and objects on the school grounds may be very accurately placed by measuring in two directions from the starting point, the school house. One might study the street location scheme of the town as an exercise. After local geography has been studied the idea can be extended to the county and state map making.

In the elementary grades the maps in the books should be ruled on the same scale as the graph paper used, varying the size of the scale according to the grade or degree of advancement of the student. In the primary grades the scale might vary from one inch to one half of an inch, and in the upper grades to at least a quarter of an inch, the standard scale used in architecture. The ruling of the texts in the elementary grades should be done by the teacher. Then with graph paper of the same scale the map making becomes valuable. Boundary lines, rivers, mountain ranges and other items studied can be placed in correct relation quickly. The child thus gets a correct viewpoint with respect to relative position, distance, size and direction.

In the intermediate grade the exercise is extended to latitude and longitude, the equator and the prime meridian being the base lines. Cities correctly located with respect to latitude and longitude enables the student to get an experience that is associated with the data, hence he works with a clearer understanding. The study of township and range lines with graph paper ruled to represent townships, with their sections of land, furnishes problems of location, form and area. Many other exercises suggest themselves to the teacher.

The object to be gained in the use of the graph paper is the accuracy of the mental picture given to the student. If the visual picture is accurate, the audile impression correctly given, and both co-ordinated with the muscular sensations arising from the actual and accurate placing of the points being studied, the work must have an educational value.

The plan also gives a chance to introduce the mathematical notion of functional relation, that is that everything depends upon some other thing. This kind of work leads to a graphic representation of soil survey, the amount of precipitation, location of drainage pipes, the yield of grain per acre, the laying out of roads and walks, the drawing of plans for houses, the cost per student per year for his education in the various grades, the relative income and expenditure in the various economic and social activities.

Not only does the above plan give economical results in certain phases of geography teaching but helps to relate the geography to arithmetic in a working way, thus making for unification in our system of knowledge.

"Mathematics is the glory of the human mind."—Leibnitz.

"Mathematics is the most marvelous instrument created by the genius of man for the discovery of truth."—Laisant.

A PLAN FOR THE STUDY OF STOCKS AND BONDS

By Wm. H. Zeigel

Experience gives the student the best insight into business forms and practices. Checks, notes, deeds of trust, warranty-deeds, chattel mortgages, quit-claim deeds and insurance policies should be handled and studied. The offices of the notary public and recorder of deeds should be visited, their functions and records examined. When this is done and the class has mastered some of the simpler things suggested, it is ready to take up the study of stocks and bonds.

The Normal Schools are in close touch with rural life. An observant person quickly sees that in farm work alone, corporations scarcely exist. Some alert student will say: "We should have corporations for agricultural purposes. Such a company could utilize the services of a skillful manager. This manager could secure experts in every line of farm production that might prove profitable where this corporation does business." We now have a setting full of suggestions for that most fundamental, difficult, and scientific occupation. Let us then proceed with the organization of a joint-stock company for agricultural purposes. I shall report this briefly, but in the main as it has been carried out in some of my Teaching of Arithmetic classes. We have followed the plan outlined in the "Corporation Laws of the State of Missouri." The procedure varies slightly in different states, and the reader would do well to consult Brown and Coffman's "How to Teach Arithmetic" and Thurston's "Business Arithmetic."

(1) A group of interested individuals in the class come together and decide to organize a corporation for agricultural purposes.

(2) These parties solicit subscriptions for the stock.

(3) Three or more persons subscribe and actually pay up 50% or more of the proposed capital stock.

(4) A meeting is then called of all subscribers and a perma-

nent organization is effected. Articles of corporation are drawn up showing:

(a) The name of the proposed corporation,

(b) The place of business,

(c) The amount of capital stock ($200,000), the number of shares (2,000), and the par value of each ($100), and that 50% of the capital has been subscribed,

(d) The names and places of residence of the several shareholders, the number of shares subscribed by each, and the number and names of the directors agreed upon for the first year,

(e) The number of years the corporation is to continue,

(f) The purpose of the corporation,

(5) All the stock holders sign this instrument and it is acknowledged before a notary public, and recorded in the office of the recorder of deeds of the county in which the corporation is located. A certified copy of the articles of incorporation is then filed in the office of our student Secretary of State, who issues a charter to our corporation, thus showing that it is legally organized and ready for business.

(6) Officers are chosen by the directors and shares of stock are issued to subscribers, and all other stock of the corporation not subscribed and paid for, at the time of organization may now be sold at par value and report be made to the Secretary of State.

In the process of organization many very interesting questions in law and parliamentary procedure arise which cannot possibly be mentioned in this brief article. But it is a most effective correlation of government and arithmetic, and in this particular case of agriculture also.

Work of Directors

Now the work of the Directors begins. They elect a Farm Manager and with him determine the policies of the company. A farm must be bought, its location as to ready markets, improvements, water, kinds of soil, crops, good roads, church and school facilities, must all be carefully weighed. With these factors in mind different properties are considered until finally a 1000 acre tract at $110 per acre is found. Then the question of improvements and live-stock for the farm is considered. There is much discussion as to whether the $90,000 that is left is sufficient to stock, erect buildings, fence and equip this farm. When at length this is deemed sufficient, the special kinds of activities, such as production of cereals, stock farming, dairying and horticulture are advocated; and experts are elected to take charge of each department of work. Then come the plotting of the farm, soil analyses, crop distribution, and the location of pasture lands.

Dividends on Common Stock

Let us assume that our enterprise is now in operation, and that at the close of the first year we cleared $8000. The directors may declare this as a dividend, or all or any part may be used as a reserve and invested in the business. If the whole is distributed among the stock holders a 4% dividend can be declared. John who bought 30 shares will receive $120; James who bought 50 shares will receive $200.

Suppose the second year we clear $14,000. It is now evident that the directors could declare a dividend of 7%. We must not fail to observe that the dividend is reckoned on the par value ($200,000) of the stock. If during the third year we clear $16,000 the Board could declare a dividend of 8%. But, if by good management and a favorable season, we can during the fourth year clear $30,000, we may then declare a dividend of 15%. The 30 shares owned by John now yield $450, and the 50 belonging to James bring him $750. Suppose Tom and Henry did not invest at first, but would now like to buy stock in our company. John sells Tom 15 shares at $200 each, and James sells Henry 25 shares at $250 per share. If again during the fifth year the company clears $30,000, another 15% dividend can be declared, and this is on the par value of the stock. The par value of Tom's 15 shares is $1,500, therefore Tom receives 15% of $1,500 or $225. John also receives a dividend of $225 on his 15 remaining shares. But Tom paid $3,000 for his 15 shares while John only paid $1,500 for his 15 shares. Tom then receives 7½% on his investment while John receives 15% on his investment. So also James and Henry each receive $375 on their 25 shares. But since Henry paid $6,250 for his 25 shares, he realizes but 6% on his investment; while for James who bought at par this is 15% on his investment.

This question then immediately arises: Is it fair that Tom and Henry should make only $7\frac{1}{2}\%$ and 6% respectively on their investments while James and John make 15% each? Uncertainty of income and principal, and priority of investment are advanced in justification. Moreover it is pointed out that it would be unfair for Tom to receive a larger dividend than John since they have the same number of shares. The same may be said of James and Henry, and the whole process for computing dividends on common stock, and per cent of income on investment has become clear.

Dividends on Preferred Stock

The directors may agree, because of their growing business, to buy an adjoining 160 acres of land which can be purchased at $125 per acre. For this purpose they may decided to issue 6% preferred stock, which they find can be sold at $125 per share. The issue will then consist of 160 shares of preferred stock, with a par value of $16,000. Suppose the Company makes a profit of $30,960. The Board of Directors will first have to pay a 6% dividend on $16,000 or $960 before a dividend can be declared on the common stock. This will leave $30,000 to be divided among stock holders owning stock whose par value is $200,000. Therefore a 15% dividend can be declared on the common stock. It is at once seen that a person owning one share of preferred stock receives a dividend of $6, but since he paid $125 for it, he receives but $4\frac{4}{5}\%$ on his investment.

Watered Stock

Finally, Mr. Ash comes along with a patented machine for making artificial fertilizers, and Mr. Bates comes with a hog cholera serum and a serum for the hoof and mouth disease of cattle, and convince our directors of the merits of their patents. Mr. Ash gives to our company the exclusive right to his patent for 750 shares of common stock; to Mr. Bates we give 500 shares of common stock for the patents pertaining to his serums. We now have 3120 shares of common stock and 160 of preferred stock. After trial the machines and serums prove valueless. The fertilizers do not enrich, and the hogs and cattle that were treated die.

Our profit for the year was $5,640. We must first pay the 6% preferred dividend on 160 shares. This leaves $4,680 as a dividend on 3,120 shares whose par value is $312,000. We can then declare a dividend of $1\frac{1}{2}$%. It will be observed that Ash and Bates have received valuable interests in our company, while we in turn have received an interest in their worthless patents; and relying on their efficacy have suffered a loss in production. Next Mr. Case has an adjoining 10 acres which experts show is rich in ore. Our directors issue 980 shares of stock to Mr. Case for this property. But the ore is refractory and cannot be separated. We have then 4,000 shares of common stock and 160 of preferred stock. We again make $30,960. After paying the 6% dividend on the 160 shares of preferred stock we have $30,000 to declare as a dividend on 4,000 shares of stock whose par value is $400,000. This dividend is $7\frac{1}{2}$%. In the case of the machine, the serums and the mining land we have issued stock for property or rights at an over-valuation. Our stock is said to be "watered."

Advantages of a Corporation

The method of organizing a company, the terms used, the computation of dividends, per cent of income, kinds of stock, and watered stock should now be clear and real.

The class will be able to enumerate and illustrate some of the advantages of a corporation:

(1) Each stock holder is financially responsible for only the par value of his stock.

(2) Opportunity for small investors to participate in great business enterprises;

(3) An easy method for the transfer of ownership;

(4) The simplicity of the bookkeeping incident to transfers and computations of dividends;

(5) Possibility of acquiring an interest in a business where one would not be qualified to assume direct control;

(6) An easy means of borrowing money;

(7) Possibility of great expansion in capital and organization;

(8) Unity of purpose and management which enlists the expert in all production.

Corporations and Partnerships

The class will be able to compare corporations with partnerships. The students will readily see how difficult it would be to carry on a great business consisting of a thousand partners, where their individual notions would interfere with any definiteness of purpose, and each by his individual acts would be able to bind the credit of the company. The stockholder in a corporation is free to sell his stock, without asking the consent of any one, while in a large partnership because of trade and influence and good-will built up, it would be very difficult to release partners or to take in new ones. When a partner dies, the firm is dissolved. A large partnership would always be in confusion; the calculations, because of the varying sums, and the varying periods of time involved, would be most complicated. Corporations for production constitute one of the greatest achievements of the centuries. Without them modern business enterprise would be impossible. Business efficiency would disappear, and the world's production would be reduced by half.

In a very similar manner bonds may be studied. This excursion into the realities of life may seem to have but little arithmetic about it, but let it be remembered that many girls and boys study for a long time the subject of stocks and bonds and when through have but the faintest idea of the subject. If this plan can enable the student not only to solve the problems, but to understand this very important subject, it has served the real purpose of all teaching in making the theory and the problem function in the concrete actualities of life.

"The progress, the improvement of mathematics are linked to the prosperity of the state."—Napoleon.

THE HISTORY OF MATHEMATICS AS AN INCENTIVE TO MATHEMATICAL STUDY

By G. H. Jamison

The teacher of high school mathematics sometimes feels that he is in a field of work very much isolated from the rest of the world. He, along with some of his students, sometimes wonders if there is any connection between mathematics and the every day life which one constantly meets. To many people mathematics seems to have had its origin in the peculiarly endowed minds of a few geniuses of the past. This paper hopes to suggest a field to which the teacher may go not only to find the proper relation which mathematics bears to life but also to get material for enlivening the interest of the pupils in the great science.

Who thinks now-a-days of teaching Caesar without maps and references to the history of the wars of that great general? Along with the translations the class constructs models of the bridges and forts used in the campaigns and studies the plans of battle. The subject is robbed of some of its abstractions by references to the customs of the people, and to their manner of living. Latin may once have been a dry, uninteresting subject with no relation to life but it is not so now. The teacher of mathematics has something to learn here. Our mathematics with its demonstration of propositions, varied now and then with original exercises and corollaries, and with the x's, y's and z's of algebra has seemed abstract and far removed from anything of life. To remove some of its abstractions, to relate it with the progress of civilization, to create an interest in it by references to its failures and struggles in development and by a study of the men who have given their lives for its truth, constitute some of the principal uses to be made of the history of mathematics in the teaching of that subject.

The history of mathematics should give the teacher a point of view in approaching the subject. There is much truth in the theory which says that "the child learns somewhat as the race has learned." Where the race has had struggles the child will probably have difficulty. It has taken the world hundreds, yes, thousands, of years to learn some of the simplest facts of our arithmetic—facts which made possible and hastened the progress of civilization. A study of the hard struggles which the race has encountered in developing its mathematics will give the teacher a patience that will not expect pupils to learn it all in a brief hour. The history of mathematics should be a sort of pedagogical instructor for the teacher. It will give him the order of development of the different branches of mathematics. He will learn that most schools of our country violate this order, while many European countries follow it. He will discover that the present plea of leading educators for the unification of mathematics is in line with the historical development. There is no need and no justification whatever for handling arithmetic, algebra and geometry as three compartmentally distinct subjects. History teaches us that these three subjects arose at practically the same stage of civilization and have advanced in close contact, constantly giving and receiving aid from each other. And the geometry which we teach last has given most aid.

Nearly every topic of study has a history which is very interesting. It may be the student is studying the ratio of the circumference of a circle to its diameter. We call this ratio π (pi, a Greek letter). In the British museum is a papyrus, known as the Ahmes manuscript, which dates back to about 2400 B. C. This papyrus states that the area of the circle is, as we would state it $(\frac{8}{9})^2d^2$ or $\frac{64}{81}d^2$. Since the area $= \frac{1}{4}\pi d^2$, it is clearly seen that the Ahmes value of π is $\frac{256}{81}$ or 3.1604. If the student is directed to I Kings 7: 23 or II Chronicles 4: 2, he will find that the people of that time thought its value to be 3. With an interesting historical background let the pupil proceed to find the value of π even more accurately than the people of long ago. The Hindu system of notation suggests an hour of great interest. Study the symbol zero. Without it, progress was well nigh impossible. It made its first appearance in India about 876 A. D. It was perhaps first written as a dot, and the Arabs, who have not yet come in contact with western civilization, still use the dot as zero. Since the dot could be easily erased by the traders, it was written as a polygon and then it degenerated into the present symbol. It was not widely used in Europe until about four hundred years ago. The first definite trace of zero among the Chinese is in the work of Tsin of 1247 A. D. Of equal interest would be a study of our own cumbersome system of measurements, logarithms, fractions, the metric system, etc. In arithmetic or geometry when the class comes to the famous theorem of Pythagoras let the pupils know something of that man, of the great school he founded, of the great society of mathematicians and their queer habits and rules. Let them know that mathematicians were sometimes killed simply for revealing the methods of solving problems. One man was drowned for stating that the sun was larger than the Pelopennesus. The reciting of these facts now and then will awaken the pupil and should introduce into the abstract work a human element of great value. This paragraph is only suggestive of the rich material which a teacher can have at hand for creating an interest in mathematics and showing that it is an outgrowth of the needs of the people.

Most of the textbooks of recent date are introducing brief historical notes and pictures of the great mathematicians. With their notes as a clue the teacher can go to any text on the history of mathematics and find valuable and interesting material. One teacher who has made large use of the historical background gives this as the result: "It captivates the wandering interest and creates a new vigor in the solution of all problems." Better than learning how to solve a problem is the desire to solve it. The taste for the subject as well as the subject matter should be in the teacher's thought as she prepares and conducts the recitation. The former attitude has been too much neglected. "I want to learn more of the subject" should be the expression read in the pupil's face when a given task is completed.

Another suggestion is that the rooms devoted to the teaching of mathematics be decorated with the pictures of leading mathematicians. These may be had without much expense. Have the class look at the pictures of Descartes or Euclid or Napier as it studies the contributions of these men. Would it not be worth while to have a picture of Sir Isaac Newton, of whom Laplace when speaking of the work which contained the principles of the calculus discovered by him, said: "It will always remain preeminent above all other productions of the human mind." One of the principles which Brandford enunciated is that we should "make the pupil gradually familiar with the great names, the interesting facts in mathematical history, the long struggles, the frequent blunders and the final successes of his ancestors in building up the science."

The history of geometry will tell us to do away at first with rigid proofs. Very briefly the development is thus stated. The early Greek geometry began with Thales about 600 B. C. His successors are Pythagoras, Hippocrates, Plato and finally Euclid of about 300 B. C. The history of these 300 years records the evolution of geometry from results based upon observation, experiment and intuition to proofs based upon induction from particular cases, the stage reached by Pythagoras. From him we trace its development to the growing strictures of proof ascribed to Hippocrates and finally to Plato who was the first to provide geometry with an array of definitions and postulates. At this point geometry fell into the hands of philosophers who took it from the purely practical field and stated the logical principles of mathematical deduction. The geometry then went through the refining stages until it was developed into the completeness we have in Euclid. Thus it is seen that history justifies the movement to rely greatly upon intuition in the early stages of the study of geometry and not to insist too early upon rigid demonstrations.

The history of mathematics will give anyone a great respect for the science which is at the basis of the world's progress. Mathematics antedates all the other familiar sciences, astronomy alone excepted. It has been termed the hand maiden of the sciences. One of the sources of its greatness is due to the fact that it is the servant of everything. It is largely if not entirely due to the aid of applied mathematics that the great war in Europe is now being waged as no war before ever has been fought. Just a few weeks ago Secretary Daniels of the Navy Department, in arranging for the personnel of the advisory board for the Bureau of Invention, wrote to the presidents of the eight leading scientific societies of the country, asking that two members be selected from each. The American Mathematical Society has selected two. Thomas

Edison the inventor says that he does not have time to learn mathematics, but he has a high priced mathematician constantly at hand. But mathematics is more than a servant. Perhaps more than any other science is it able to stand alone. It is a world unto itself containing truth and beauty, and when you have mastered the secrets of nature, some principle from mathematics is used in applying or giving to the world this secret. Says a man prominent in the educational world, "One of nature's demands in which she is inexorable is a study of higher, the highest mathematics. The interpretation of her laws requires it."

If we take as the standard of appreciation of a science the application of it to daily life, mathematics will not suffer. Samuel G. Barton of the University of Pennsylvania selected from the Encyclopedia Britannica, the 11th edition, the subjects which have required the symbols of infinitesimal calculus in their treatment. The list has one hundred four headings. A few of them are as follows: aether, bridge, calculating machine, chemistry, clock, illumination, light, lens, lubrication, map, ship-building, sky, steam engine, sun, tide and measurement of time. If a lower subject had been selected such as trigonometry the list would have been far greater. Only about one fourth of the headings are pure mathematics.

If the teacher finds the recitation hour growing dull and the pupils sleepy and wants to add zest to study recite some of the incidents suggested by these topics: The Story of the Pythagoreans and the five pointed star as the emblem of their fraternity, the three famous problems of antiquity, the inscription on Plato's door, the royal road to geometry, the story of Cardan's theft, Pascal's study of geometry when forbidden to do so by his father and his writing a treatise on conic sections when fifteen years of age, the great contests in problem solving in which Descartes and Newton participated, the death of De Moivre, the story of the invention of the symbols of arithmetic, etc., etc. Of more than passing interest are some of the mathematical experiences of men who have figured most prominently in the history of our country. Washington was very fond of mathematics and remained in school two years longer than he had intended in order to study that subject. When Lincoln left Congress in 1849 he felt a lack in himself of the power of close and sustained reasoning and to remedy this defect and to get a better education he applied himself to the study of logic and mathematics. General Grant was very strong in mathematics and he never was willing for any one else to work his problems for him. Jefferson kept up the study of mathematics as long as he lived and a biographer says that "he delighted in applying its principles to anything and everything." When President Garfield was a boy in school he wrote out an original proof of the Pythagorean theorem which is quite simple. Webster speaking in 1828 of pure mathematics says that "it is ignorance only which can speak or think of that sublime science as useless research or barren speculation."

The student of music goes to his history and learns about the masters of music. The artist knows the Raphaels and De Vincis. One would not think of studying literature, without first learning something of the lives of the makers of literature, and why should the student of mathematics be left in ignorance of the great history which has attended its development? This question becomes emphasized and must be answered in view of the great light and interest which the history of mathematics gives to the study of it.

"God geometrizes continually."—Plato.

"No subject loses more than mathematics by any attempt to dissociate it from its history."—Glaisher.

THE VALUE OF A LIFE SETTING TO ALGEBRA PROBLEMS

By Byron Cosby

In a recent publication Dr. Charles Hubbard Judd, states that in eleven suburban high schools surrounding Chicago, thirty per cent of the number of students enrolled in the first year high school algebra either withdraw from the subject or fail to pass. While the number is no higher than the loss in some other high school classes, it is evidence that our present course in first year algebra is not well suited to the first year high school student. Our present first year course in high school algebra is too formal. The course is so arranged that the student may have a chance to get great technique in the formal algebra, but is not taught to apply the principles readily and accurately.

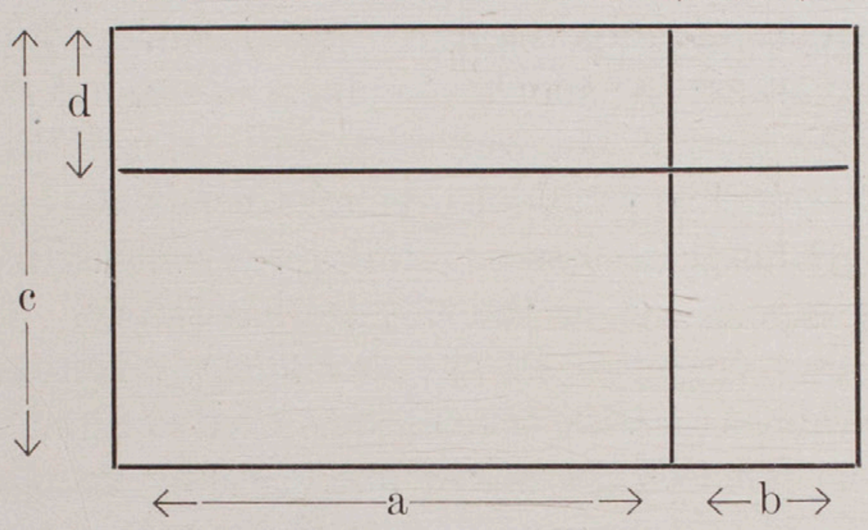
At the present time there is much discussion in the newspaper, professional magazine, recent educational publication and teachers associations as to the advisability of continuing our present type of algebra in the first year high school. The history of the development of algebra would probably place algebra after the geometry, as a glance at Greek history will show. Also there seems to be a dissatisfaction among the mathematics teachers as is evidenced by the new texts, the development of new methods of presentation; as the graphical formula rather than the equation, applied or real problems and other schemes that are suggested. The most enthusiastic supporters of algebra feel that the material of the first year high school algebra should be changed, rearranged, approached from another angle or presented in a different form.

Algebra is an economic symbolism. As a type of notation it is valuable. In the beginning the student must learn to translate from his thinking language over into the symbolic language of algebra. Before a translation can be made the student must know something about the theme under discussion and must understand the notion in the terminology of the language that he speaks, not that of the teacher or the book. When he has a working knowledge in his language, he will have a chance to work the material over into the symbolism of algebra. Therefore, since a student can function better in those things that he is acquainted with, it is necessary for economic reasons to find a concrete or physical way of approach to the algebraic problem. If the problem can be associated with some notion or activity that the student is familiar with, or can easily interpret from previous experiences, it is the more readily understood. To start with an assumption that may be obvious or not, and then by logical deductions arrive at a conclusion may give a pretty piece of work, but not necessarily attractive to the student.

Believing that familiarity with the material gives greater ease in translation from verbal language to algebraic symbolism, this paper attempts to give some individual illustrations in which a life setting or a familiar evaluation may be placed upon the problem. This notion does not limit the field of algebraic endeavor for algebraic problems have not only the possibility of being extended to the solution of some situation in the world's activity, but also as a question of, or a part of mathematical truth. If algebra is to be taught for its cultural value, that is as a part of the world's great body of truth, used in social and economic organizations, and as a tool for material construction and mechanical adjustment it is essential that the setting of the notion be as closely related to life as possible.

In introducing the idea of area the problem should be associated with the arithmetical situations of the rectangle and the triangle. In lettering the figure it is best to use the first letter of the word describing the position spoken about, as b for the word base, h for height and A for area. From the numerical problems of arithmetic it is easy to pass to the algebraic formula for area of the rectangle or triangle. In like manner the arithmetic situations in distance, rate, and time or in the interest problem involving the principal, rate, time and interest give familiar approach to the problem. The formula $A = (1 + \frac{r}{100})^n P$ has little significance because of its compactness, but when the arithmetic or life setting that the student has may be translated over into this formula, and the formula used in finding the value of his building and loan investment, or the value of his father's life insurance, he sees a good reason for studying logarithms.

Is it better to make the positive assumption and state dogmatically that like signs in multiplication give plus and unlike signs give minus or to use the following picture? Assuming that the student has had enough arithmetic to understand that the number representing length multiplied by the number representing width gives the numerical value of the area, use $(a+b)$ for the



length and $(c-d)$ for the width. $(a+b)(c-d) = ac+bc-ad-bd$. It is easily noticeable that the area of $(a+b)$ or length times $(c-d)$ or breadth is the total area with the upper strip taken away, and since we have always designated taken away by a minus sign, even in arithmetic, the area ad and bd will be prefixed by a negative sign. Then if we suppose c to be zero, we see that we have $(a+b)(-d) = -ad-bd$. This is not offered as a proof that plus by minus gives the minus sign, but as a means of approach to the notion that is within the field of the child's experience. Which is the better the assumption or the approach suggested?

The study of graphical representation by rectangular co-ordinates may be easily approached in map making or map drawing. The map in the book may be ruled to any scale and a sheet of paper on which the map is to be made ruled to the same scale. The lower margin and the left hand margin may serve as the direction lines, which may be called the axes. From the map drawing to a study of latitude and longitude, with the prime meridian and the equator as the base or direction lines, one may advance to a closer study of rectangular co-ordinates. The words north and east may be replaced by the plus signs and south and west replaced by the minus signs. As a further extension of the idea the location of the streets and the numbering of the houses from certain base streets in town, together with the problem in land surveying, in which the principal meridian and the base line are the axes, an easy transition may be made to the generalization of rectangular co-ordinates, together with the technical or particular names that we have assigned to the properties of the graphical situation.

In the more advanced algebra do not introduce trigonometric functions by asking the student to plot $y=\sin x$, but rather have him study the laws of the pendulum, until he sees that the time of a freely swinging pendulum remains constant, but that the amplitude gradually decreases. In the study it may be necessary to try the experiment in the physics laboratory. Then recall that the ratio decreases as the denominator increases, if the numerator cannot increase beyond some fixed constant value, have him plot the function $y=\sin x/x$ and then by translation back and forth see if he does not understand the graph, and see if he does not understand it in terms of his laboratory or life experience.

Many objections have been advanced against the clock problem in algebra, but it is an easy approach to an important problem in astronomy, and an approach within the field of the child's experience, to a study of the actions of the planets. The earth travels eleven times as fast around the sun as Jupiter, giving the same problem, with almost the same numbers as in the clock problem. The same problem is found in the study of the moon's phases. It is obvious that such an approach to a problem will mean more than the abrupt use of the equation, because the equation within itself does not simplify the work. The dominant thought in algebra is functional thinking. The question always is, the magnitude that we must measure depends in what way upon some magnitude that we know how to measure. This paper is not an argument for the "applied or real" problem, because such a problem does not necessarily have any significance in the acutal life of the child, or even in the practical problems of physics.

This method of presentation of algebraic problems, of which only a few illustrations have been given, demands resourcefulness and initiative on the part of the teacher, and more or less intimate understanding of the elementary principles of many of the activities of life, but it is not as wasteful as the old formal way and is more natural. At least it is worth thinking about, as the projection of an idea into another plane or another setting, or a translation may give greater insight into its possibilities. If algebra can be translated into terms that a student can understand and then translated back into a concise formula, it will be worth while as a high school subject because of its economic notation.

"In the high branches of the science of pure mathematics lies the true sublime of human acquisition."—Webster.

"All scientific education which does not commence with mathematics is, of necessity, defective at its foundation."—Compte.

ANALYTIC GEOMETRY IN OUR HIGH SCHOOLS

By Wm. H. Zeigel

Algebra and geometry were always taught together in early times. Both primitive algebra and primitive geometry are found in the Ahmes papyrus. In the elements of Euclid there is the fusion of pure geometry, geometrical algebra, and theory of numbers. The "Summa" of Pacioli (1494) contained all of the branches of mathematics. The development of these branches has been such as to give each a content of its own. Geometry deals with form, but form may be arrived at by algebraic processes, or deduced from algebraic relations. So also algebra deals with functional relations, but these relations may be arrived at geometrically or pictured geometrically. Thus each branch may help to explain and clarify the other. But all attempts of modern times to bring these two subjects, with their vast accumulations of important theorems and processes, into a perfect fusion have usually met with ill success. This failure may be partially accounted for by the fact that the work of elementary algebra, though dealing with simple functions, is much more easily mastered than is the logic involved in dealing with form as set forth in geometry. The first can be comprehended by the child mind, the other often requires a mature mind. So there seems to be a period of separate development required, where each subject shall receive special treatment, where a constant effort is made to bring about all possible natural co-ordinations.

Now, since these two branches of mathematics do throw so much of light on each other, is it not appropriate to ask whether or not there is a place in the high school curriculum, where these two subjects can be brought together? May it not be well to give to the great mass of our children, who in general will not be permitted or inclined to continue their studies beyond the high school, a chance at least of seeing these two subjects to their best advantage, as revealed in analytic geometry?

To be sure I would not urge this upon all the children of our high schools. In our complex civilization the high school has become a necessity. More and more, large numbers, if not quite all of our children, will attend our high schools. The courses offered must ultimately be of a nature to make our boys and girls efficient in all occupations and professions of life. The school must exist for the children, and prepare each to do better the work he is fitted to do. Thus our courses will come to be largely elective. Even algebra and Euclidean geometry may become elective courses. So also for those who like mathematics, some elementary phases of analytic geometry might well constitute an elective course in our four year high schools. To produce mathematicians, it is not desirable to compel children, that have shown a lack of capacity for the subject, to continue its study. To do so wears out the teacher and discourages the pupil. It is desirable however to give to every boy and girl, who is mathematically inclined, a glimpse into the higher and more generalized processes, so that he may be able to judge the subject of his own accord. Analytic geometry is so different from ordinary geometry in its fascinating power, that were it taught in our high schools, it would no doubt cause many of mathematical ability to discover themselves.

But granted that it might be introduced for the purposes and under the conditions mentioned above, (1) what would be the preparation required in algebra, Euclidean geometry, and trigonometry? (2) Can analytic geometry be understood by high school students? (3) What distinct advantage would it give?

In answer to question (1), it may be said that a minimum of one and one-half years' work in algebra should be given; that from a year to one and one-half years should be devoted to plane geometry, together with the elements of trigonometry; but that analytic geometry should take the place of the one-half year elective course usually offered in solid geometry, and for students, who are especially interested in mathematics, analytic geometry might well take the place of the one-half year of high school arithmetic that is offered in many of our schools. Thus with a maximum of one year of analytic geometry we need not have more than four units of high school mathematics. The analytic geometry of the straight line and circle would be quite sufficient for our purposes, and the equations of these curves should be referred to rectangular axes only. By examining the theoretical work of the first one hundred pages of Ashton's Analytic Geometry that uses rectangular axes, we find but two theorems of Euclidean geometry used. Namely: The Pythagorean theorem, and the one regarding the ratio of corresponding sides of similar triangles. The first is needed to find the distance between two points, the second to divide a line in a given ratio, and to find the equation of a straight line through two points. But both of these theorems are familiar to students from their study of arithmetic. However, from trigonometry we need to know the meaning of a tangent of an angle as soon as we come to the equation $y-y^1=l(x-x^1)$. We also need to know that $\tan \Theta=\frac{l_1-l_2}{1+l_1l_2}$, when we find the angle between two lines, or when we find the equation of one line that makes a given angle with another line. Also we need to know the significance of sine and cosine of an angle in finding the normal form of the equation of a straight line; also the fact that $\sin^2x+\cos^2x=1$, which is used in reducing the general equation of a straight line to the normal form. But this is the extent of our geometrical and trigonometric requirements, and it is evident that these trigonometric functions are mainly dependent upon the two geometric theorems enumerated. Then it would appear that with a proper selection of material from plane geometry and trigonometry, that one year devoted to these subjects would be sufficient to fit those who are apt in mathematics for the consideration of the simpler notions and problems of analytic geometry. To be sure, the problem work of the text which I have cited might require more theorems than the ones mentioned; but it is also to be remembered that it is possible to obtain the theorems of our Euclidean geometry analytically. No doubt a special text should be prepared for high school use, embodying a homogeneous course well worked out.

Question (2), "Can analytic geometry be understood by high school pupils?" To answer this question we should recall the limited number of geometric and trigonometric theorems needed; also the fact that there is no algebra required beyond the quadratic equation, the discriminant being an important notion. Furthermore the graphical work of algebra will serve as a good introduction to the subject. It is also significant that Newton, perhaps the greatest mathematician of all times, studied and mastered Descartes' analytic geometry before he ever studied Euclidean geometry. Moreover Descartes' geometry was perhaps made obscure with the idea of rendering it difficult, so that others might not acquire this new instrument of investigation. Also it is a well known fact, that Newton, when he took his bachelor's degree, was deficient in Euclidean geometry. Barrow, his teacher, recognized it. Nevertheless, Newton had mastered analytic geometry, and had at this time invented fluxional calculus, and had discovered the binomial theorem. The idea of introducing analytic geometry into the high school is no mere fanciful theory. In the work of the eleventh and twelfth years, France, Germany, Austria, Sweden, Denmark, Switzerland, and Roumania have courses in both analytic geometry and calculus. The student who has completed twelve years' work in these countries has studied more mathematics than is offered in any of the secondary schools of the United States. Also analytic geometry is taught in a few of the high schools in the state of Indiana. Therefore we seem justified in the view that the subject is not too difficult.

Question (3), "What distinct advantage will the study of analytic geometry give?" It will acquaint the student with a subject that has revolutionized the methods of all mathematical investigations. Euclidean geometry had reached the limit of its possibilities. In order to solve a problem propounded by Pappus, Descartes was led to the invention of analytic geometry. The quadrature of the circle, and the rectification of curves in general were beyond the powers of this ancient geometry. Euclidean geometry is a tedious process. It requires a special procedure for each problem considered. Analytic geometry, however, deduces general principles by which any supposed property can be shown true or false. As an illustration of my meaning, suppose a point P has co-ordinates (2, 3) and that a third straight line has the intercepts 1 and 2 on the x and y axes of reference. Find the distance of the point P from this third straight line. In Euclidean geometry this would be a special problem, and though it might have a general solution, the generalized result would stand out as an isolated fact. But how different it is in analytic geometry.

The equation of the straight line is

$$\frac{x}{1}+\frac{y}{2}=1 \qquad \text{or}$$

$$2x+y-2=0.$$

In the normal form this becomes

$$\frac{2x}{\sqrt{5}}+\frac{y}{\sqrt{5}}-\frac{2}{\sqrt{5}}=0.$$

By analytic geometry the distance, d of a pt(x^1, y^1) from a straight line $x\cos\phi+y\sin\phi=p$ is $d=x^1\cos\phi+y^1\sin\phi-p$.

$$\therefore\ d=\frac{2(2)}{\sqrt{5}}+\frac{3}{\sqrt{5}}-\frac{2}{\sqrt{5}}=\frac{5}{\sqrt{5}}=\sqrt{5}.$$

Thus in analytic geometry we have in the normal form of the equation of a straight line, the means of solving any problem of this character. Scores of similar and even more striking illustrations might be given. Let us take one other illustration. About the intersection of two orthogonal straight lines as center a circle of radius 5 is decribed. Show that the line that has intercepts $8\frac{1}{3}$ and $6\frac{1}{4}$ on these lines, is tangent to the circle. By analytic geometry the equation of the line referred to the orthogonal lines as axes is

(1) $3x+4y=25$.

The equation of the circle is

(2) $x^2+y^2=25$.

If we put (1) in the slope form, we have

$(1)^1$ $y=-\frac{3}{4}x+\frac{25}{4}$.

Now by analytic geometry $(1)^1$ is tangent to (2) if $b=r\sqrt{1+l^2}$ where $b=\frac{25}{4}$, $l=-\frac{3}{4}$, $r=5$. Then $\frac{25}{4}=5\sqrt{1+\frac{9}{16}}=5\cdot\frac{5}{4}=\frac{25}{4}$.

$\therefore$ $(1)^1$ or (1) is tangent to (2). It is worthy of notice that the method used in the preceding illustration could also have been used here instead of the plan employed. Did time permit, it would be interesting to compare this with the Euclidean solution of the problem. Moreover, the method used here for the circle is a special case of a more general method applicable to all conics. Thus we see the problems of the Greeks were isolated. The solution of one did not prepare the way for another. But in analytic geometry, from some geometrical property, we obtain an equation involving x and y, the running co-ordinates of the point, and no other unknowns. This equation contains every property of the curve, and it can be deduced without any further reference to geometrical constructions. So it is that multitudes of problems and theorems become special cases of more general theorems, and a mass of detail that burdens and confuses the mind is eliminated.

We have already pointed out the relation between the graphical work of algebra and analytic geometry. Let us give one other illustration of the vital connection between this new geometry and algebra. We have learned in algebra, that if we solve a quadratic equation in x and y with a linear equation involving the same variables, that we always get two solutions, real, coincident or imaginary. We have seen that the graphical representations of the real solutions are the coordinates of the two distinct points of intersection; but that when the solutions are coincident the secant line has become a tangent line, and therefore the condition for tangency is the condition for equal roots, when the quadratic equation and the linear equation are solved simultaneously. For instance let

(1) $x^2+y^2=r^2$ be the equation of a circle, and
(2) $y=lx+b$ be the equation of a straight line.

If, when we solve (1) and (2) simultaneously, we take the condition for equal roots, we obtain a relation between the constants which is the condition for tangency. Thus we have

$$(1+l^2)\,x^2+2\,l\,b\,x+(b^2-r^2)=0.$$

Now by algebra the condition for equal roots is

$$(2\,l\,b)^2-4(1+l^2)\;(b^2-r^2)=0$$

or

$$b=r\sqrt{1+l^2},\text{ or } b=-r\sqrt{1+l^2}.$$

$\therefore$ there are two tangents to (1), and their equations are

$$y=l\,x+r\sqrt{1+l^2} \text{ and } y=lx-r\sqrt{1+l^2}.$$

Likewise in a multitude of ways algebra reveals geometrical relations, and geometry visualizes algebra. Finally, we observe that analytic geometry by means of equations of curves and surfaces, makes calculus, that most powerful instrument of research, possible. The difficult problems of the ancients yield up their secrets willingly to this marvelous means of investigation. So too, the idea of functionality runs like a thread through the whole of analytic geometry, obtruding itself with such force upon the mind of the learner, that he cannot help but get some grasp of variation, and its underlying significance in all the problems of life.

Analytic geometry is basic in all new methods of mathematical research. The simplicity of its content, the history of the subject, the interrelations of algebra and geometry, the generality of its processes, its enlarged view of functionality, its fundamental entrance into calculus, all lead us to desire its introduction as an elective course into our four year high schools.

"If the Greeks had not cultivated conic sections, Keplar could not have superseded Ptolemy."—WHEWELL.

"The art of measuring brings the world into subjection to man: the art of writing prevents his knowledge from perishing along with himself: together, they make man—what Nature has not made him—all-powerful and eternal." —MOMMSEN.

SETS OF ORTHOGONAL FUNCTIONS AND THEIR OSCILLATION PROPERTIES

By Chas. A. Epperson

A set of functions $f_1(x)$, $f_2(x)\ldots.$ is said to be orthogonal in the interval (a, b) if $\int_a^b f_i(x)\cdot f_j(x)dx=0$ (i not equal to j) and if $\int_a^b f_i^2(x)\,dx$ is different from 0. In the memoirs of Sturm and Liouville (Liouville, Journal de Mathematiques: Vol. I-II, 1836), two classes of theorems are found concerning sets of orthogonal functions. The first deals with the number of sign-changes in $f_n(x)$, and the second with the number of sign-changes in a polynomial $c_1f_1(x)+c_2f_2(x)+\ldots\ldots\ldots c_nf_n(x)$. The functions of the orthogonal sets with which they deal are solutions of a differential equation containing a single parameter.

The question arises as to whether these two classes of theorems are consequences of the mere orthogonality of the function sets, or whether the differential equation is a necessary condition for them. We shall presently see that orthogonality alone is not a sufficient condition for the oscillation theorems in question; but that with the addition of the hypotheses of the non-vanishing of a certain set of determinants it becomes so.

In sets of orthogonal functions, $f_1(x)$, $f_2(x)$, $f_3(x)\ldots f_n(x)$ as just defined, it is frequently found that $f_i(x)$ has one more sign-change than f_{i-1} in the interval* for which the functions are orthogonal. That this is not a consequence of orthogonality alone is seen by the following example, since in it $f_0(x)$ does not vanish, $f_1(x)$ vanishes once, and $f_2(x)$ also vanishes but once.

The example is:

$$
\begin{aligned}
f_0(x) &= 1 & & \\
f_1(x) &= 1+27\left(x-\tfrac{1}{3}\right) & 0<x<\tfrac{1}{3} \\
&= -5+18x & \tfrac{1}{3}<x<\tfrac{1}{2} \\
&= 13-18x & \tfrac{1}{2}<x<\tfrac{2}{3} \\
&= 1 & \tfrac{2}{3}<x<1 \\
f_2(x) &= 1 & 0<x<\tfrac{1}{3} \\
&= -5+18x & \tfrac{1}{3}<x<\tfrac{1}{2} \\
&= 13-18x & \tfrac{1}{2}<x<\tfrac{2}{3} \\
&= 1-27\left(x-\tfrac{2}{3}\right) & \tfrac{2}{3}<x<1
\end{aligned}
$$

*Interval will be taken to mean open interval.

and the interval or orthogonality is (0, 1). Our investigation shows, however, that a sufficient condition for the sign-changes observed is the non-vanishing of the determinants:

$$\begin{vmatrix} f_0(x_0), & f_1(x_0) & f_2(x_0) & \ldots\ldots & f_n(x_0) \\ f_0(x_1) & f_1(x_1) & f_2(x_1) & \ldots\ldots & f_n(x_1) \\ \vdots & \vdots & \vdots & & \vdots \\ f_0(x_n) & \ldots & \ldots & \ldots & f_n(x_n) \end{vmatrix} \quad \begin{matrix} x_i \text{ is different} \\ \text{from } x_j \\ \\ n = 1, 2, 3, \ldots \end{matrix}$$

A set of orthogonal functions satisfying these conditions we will say has the property "D".

Let us notice also that the non-vanishing of this determinant is the necessary condition for the possibility of making a linear combination of $n+1$ functions pass through $n+1$ points, at least one of whose ordinates is not zero. Thus the condition for interpolation is the condition for the observed sign-changes.

With this introduction we may now state

THEOREM I. **In a set of orthogonal functions $f_0, f_1, \ldots f_n$ having the property "D", f_n has not more than n sign-changes in the interval of orthogonality.**

For, by hypothesis, the determinant

$$\begin{vmatrix} f_0(x_0) & f_1(x_0) & \ldots\ldots & f_n(x_0) \\ f_0(x_1) & f_1(x_1) & \ldots\ldots & f_n(x_1) \\ \vdots & \vdots & & \vdots \\ f_0(x_n) & f_1(x_n) & \ldots\ldots & f_n(x_n) \end{vmatrix}$$

does not vanish. If f_n has $n+1$ sign-changes, set $x_0, x_1, \ldots . x_n$ respectively equal to the $n+1$ values of x at which $f_n(x)$ changes sign. Each element of the last column is thus seen to be zero and the determinant vanishes, which contradicts our hypothesis.

THEOREM II. **If $F_n(x)$ is a linear combination of continuous orthogonal functions, as $F_n(x) = c_0f_0(x) + c_1f_1(x) \ldots . c_nf_n(x)$ and if the set $f_0(x), f_1(x) \ldots\ldots f_n(x)$ has the property "D", then $F_n(x)$ has not more than n zeros in the interval of orthogonality.**

If $F_n(x)$ has $n+1$ zeros call the points at which it vanishes, $x_0, x_1, x_2 \ldots x_n$. Substitute these points in the equation for $F_n(x)$. We then have $n+1$ homogeneous equations in the $n+1$ coefficients c_j.

Hence the determinant

$$\begin{vmatrix} f_0(x_0), & f_1(x_0) \ldots\ldots\ldots & f_n(x_0) \\ f_0(x_1) & f_1(x_1) \ldots\ldots\ldots & f_n(x_1) \\ \vdots & \vdots & \vdots \\ f_0(x_n) & f_1(x\ ) \ldots\ldots\ldots & f_n(x_n) \end{vmatrix}$$

must vanish, which contradicts the assumption that the determinant is different from zero.

THEOREM III. **If** $F_n(x)$, **as defined in theorem II has** n **roots in the interval of orthogonality, it changes sign at each.**

To show that $F_n(x)$ actually changes sign and does not vanish without changing sign at any of its n roots, we will suppose that at one point, say r_c, is a root of even order. For simplicity suppose that $n=3$. Then $F_3(x)$ can have only 3 roots in our given interval by Theorem II. Suppose that r_2 is the double root, and that F_3 is positive between r_1 and r_2 and between r_2 and r_3. Let us now take a function

$$F_2(x) = c_0f_0(x) + c_1f_1(x) + c_2f_2(x)$$

and make F_2 vanish at r_1 and r_3 and equal to -1 at r_2. Suppose now that the lesser of the two maxima of F_3 between r_1 and r_3 is b, and that the minimum value of F_2 is $-p$. Multiply F_2 by $b/2p$. Call the resulting function, F_{2h}. F_{2h} is negative between r_1 and r_3 and less than or

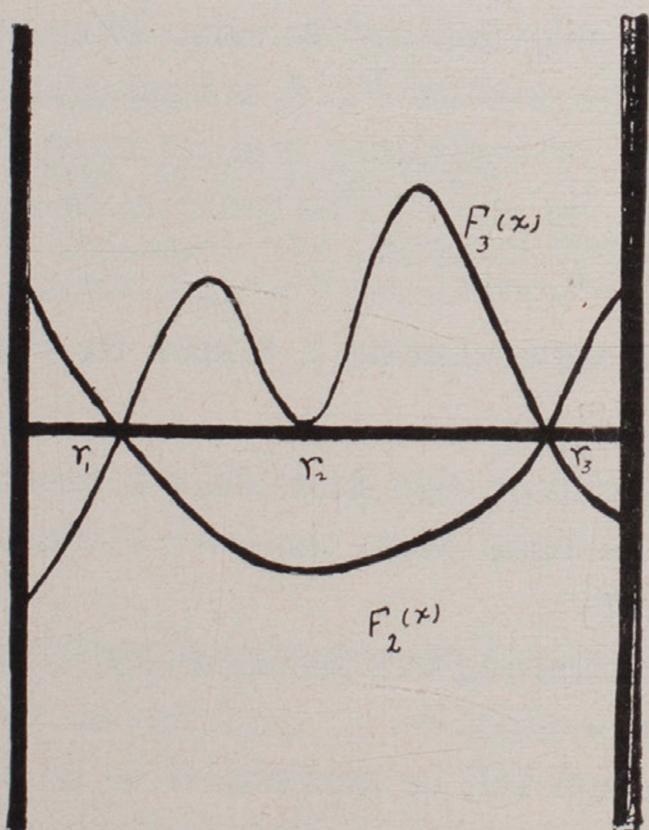


Fig 1

equal in absolute value to b/2 everywhere in that interval. Now F_3 is greater than or equal to b at some point in each of the intervals (r_1, r_2) and (r_2, r_3) and is zero at r_2. Adding F_3 and F_{2h} we get a function F_{3g} which vanishes at r_1, between r_1 and r_2, between r_2 and r_3 and at r_3. But F_{3g} is of the form $F_{3g}=k_0f_0+k_1f_1+k_2f_2+k_3f_3$, where the k's are constants, and by Theorem II cannot vanish more than three times in the given interval. Hence r_2 must be a change of sign and not a double root.

Our proof is for the case $n=3$, and shows the impossibility of one vanishing point without change of sign. It applies however in all essentials to any n and to showing the impossibility of any number of vanishing points without sign-change.

THEOREM IV. **If any function is orthogonal to the set** $f_0\ f_1, \ldots . f_{n-1}$, **having the property "D", it changes sign at least** n **times in the interval of orthogonality.**

Suppose f_k has less than n sign-changes, say $k<n$. Consider $F_k(x)=c_0f_0(x)+c_1f_1(x)+\ldots\ldots . c_kf_k(x)$. By Theorem II and the hypothesis that the set $f_0, f_1, \ldots\ldots f_k$ has the property "D", we can make F_k change sign at exactly the k points at which f_k changes sign and in addition can make it have one value different from zero in common with f_k. The product $F_k \cdot f_k$ is then everywhere positive in the interval of orthogonality and its integral taken over that interval cannot be zero. This contradicts our orthogonality hypothesis.

COROLLARY I. f_n **changes sign exactly** n **times in the interval of orthogonality.**

By Theorem I, f_n does not change sign more than n times. By Theorem IV, f_n must change signs more than $n-1$ times. Hence f_n must change signs n times.

We shall now speak of approximating to a function G(x) by a linear polynomial $c_0f_0(x)+c_1f_1(x)+c_2f_2(x)+\ldots\ldots c_nf_n(x)$ in the functions $f_i(x)$. This approximation will be understood to be in the sense that the integral of the square of the error is a minimum, that is

$$\int_a^b \Big[G(x)-c_0f_0(x)-c_1f_1(x)\ldots\ldots c_nf_n(x)\Big]^2 dx$$

is less than for any other system of coefficients k_i. It is well known* that upon this assumption the c_i are given by the formulæ

$$c_i = \int_a^b G(x) f_i(x) dx \Big/ \int_a^b f_i^2(x) dx.$$

With this definition of approximation we will state, leaving the proof to the reader, the following

THEOREM V. **If $G(x)$ is continuous and is approximated to by the continuous function $A_n(x) = c_0 f_0(x) + c_1 f_1(x) + \ldots\ldots$ $c_n f_n(x)$ where the set $f_0, f_1, f_2, \ldots f_n$ has the property "D", the remainder $E(x) = G(x) - A_n(x)$ is orthogonal to $f_0(x)$, $f_1(x) \ldots f_n(x)$, hence by theorem IV changes sign at least $n+1$ times and $A_n(x)$ crosses $G(x)$ at least $n+1$ times in the interval of orthogonality.**

The orgonthal functions in common use are the sine and cosine functions, the Legendre Polynomials and the Bessel functions. The Bessel functions themselves are not orthogonal but become so when multiplied by $\sqrt{x}$. It has been shown† that each of these sets of functions possess the property "D", and that the $A_n(x)$ of Theorem V may be a trigonometric series, a sine series, a cosine series, a series of Legendre Polynomials, or a series of Bessel Functions.

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*See for instance, Bocher; An Introduction to Integral Equations, page 55.

†Epperson: The Oscillation Properties of Certain Sets of Orthogonal Functions, Masters Thesis, University of Missouri, 1914.

"There is nothing so prolific in utilities as abstractions."—FARADAY.

A TEACHER'S LIBRARY

By G. H. Jamison

It is a well established principle that the teacher should know more than the subject he teaches. There is hardly any branch of mathematics which does not throw some light on the teaching of high school mathematics. The study of subjects beyond the immediate ones taught helps to get a general view of what is below. It is like climbing the mountain to see the whole landscape as a panorama. This study of the higher branches of mathematics enables one to look at high school mathematics as a related whole. It gives a better view point for teaching each subject. There is a very vital connection between algebra, geometry and trigonometry. The study of analtyic geometry is of invaluable help in showing that connection. The Germans, Austrians and French recognize this in the high requirements to which they hold their teachers. If the teacher of mathematics is to be successful he must read subjects beyond those he teaches. This can be done in preparation for his work. It can be done in the hour he has for his own personal development. I am suggesting below a list of books which I know will be helpful and some of them at least should be found in a teacher's library.

For pedagogical help these are good:

1. The Teaching of Elementary Mathematics by Smith. The Macmillan Co., Chicago. This is devoted largely to the historical development of arithmetic, algebra and geometry and it indicates the method of approach in the teaching of each.

2. How to Teach Arithmetic by Brown and Coffman. Row, Peterson and Co., Chicago. Also The Teaching of Arithmetic by Stamper. American Book Company, Chicago. Also The Teaching of Arithmetic by Smith. Ginn & Co., Chicago. These are the best texts in the field on the teaching of arithmetic.

3. The Teaching of Mathematics by Young. Longmans, Chicago. This is perhaps the best text written on the teaching of high school mathematics. It considers the value in the study of mathematics and the different modes and methods. In it are

chapters of considerable length devoted to arithmetic, algebra and geometry.

4. A History of the Teaching of Geometry by Stamper. Teacher's College Record.

5. The Teaching of High School Mathematics by Evans. Houghton, Mifflin & Co., Chicago.

6. The Teaching of Algebra by Nunn. Longmans, Green & Co., Chicago.

7. The Teaching of Mathematics in Prussian Schools by Young. Longmans, Green & Co.

8. Special Methods in Arithmetic by McMurry.

For reading along the line of high school mathematics and at the same time for books containing much more advanced material the following are suggested:

1. A Course in Mathematics Vol. I. by Woods and Bailey. Ginn & Co., Chicago. This book introduces much work from theory of equations, analytic geometry and calculus with a great deal of reference to the work of algebra and trigonometry.

2. Monographs on Modern Mathematics. Longmans, Chicago. This book has eleven chapters written by leading mathematicians of our country. It contains some material which is beyond the complete comprehension of students of undergraduate rank but it is highly stimulative of endeavor and progress.

3. Mathematical Recreations and Essays by Ball. Macmillan.

4. A History of Mathematics by Ball, and also one by Cajori. Macmillan.

HENRY WARD BEECHER AND MATHEMATICS

A page from the biography of America's greatest preacher will give the thoughtful teacher many lessons to take with him into the classroom. Mr. Beecher attributed much of his power as a speaker and defender of his opinion to his early mathematical training. He claimed that it was Latin and Mathematics which gave him the power of study. His teacher at the Mt. Pleasant Preparatory School was W. P. Fitzgerald. Let us read very thoughtfully these remarks from Beecher about his teacher. Says Beecher: "He taught me to conquer in studying. There is a very hour in which a young nature, tugging, discouraged, and weary with books, rises with the consciousness of victorious power into masterhood. For ever after he knows that he can learn anything if he pleases. It is a distinct, intellectual 'conversion'.

"I first went to the blackboard, uncertain, soft, full of whimpering. 'THAT LESSON MUST BE LEARNED,' he said in a very quiet tone, but with terrible intensity, and with the certainty of fate. All explanations and excuses he trod under foot with utter scornfulness. 'I want that problem; I don't want any reason why you don't get it.'

"'I did study it two hours.'

"'That's nothing to me: I want the lesson. You need not study it at all, or you may study it ten hours—just to suit yourself. I want the lesson. Underwood, go to the blackboard.'

"'Oh, yes, but Underwood got somebody to SHOW him his lessons.'

"'What do I care how you get it? That's your business. But you must have it.'

"It was tough for a green boy, but it seasoned him. In less than a month I had the most intense sense of intellectual independence and courage to defend my recitations.

"In the midst of a recitation, his cold and calm voice fell upon me, 'No!' I hesitated, stopped, and then went back to the beginning, and, on reaching the same spot again 'No!' uttered with a tone of perfect conviction, barred my progress. 'The next!' And I sat down in red confusion. He, too, was stopped with 'No!' but went right on, finished, and as he sat down, was rewarded with, 'Very well.'

"'Why,' whimpered I, 'I recited it just as he did, and you said 'No.'

"'Why didn't you say 'YES,' and stick to it? It is not enough to know your lesson. You must KNOW that you know it. You have learned nothing till you are SURE. If all the world says 'No,' your business is to say 'Yes,' and to PROVE it.'"