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FIRST DISTRICT NORMAL SCHOOL

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FOREWORD

To the Teachers of Missouri:

The Mathematics Department of the First District Normal School encouraged by the reception given to Mathematics Bulletin, series number one, now has the pleasure of offering to you Mathematics Bulletin, series number two, with the hope that it may prove equally interesting and suggestive.

BUILDING AND LOAN ASSOCIATION EXPLAINED

By Wm. H. Zeigel

Building and Loan Associations originated in England in 1789. They spread to Germany and continental Europe long before they were known in America. The first one organized in the United States was at Frankford, Pa., in 1831. Since that time there has been a remarkable growth of these institutions both in numbers and in capital stock. In the United States there were 5838 of these associations in 1893. In the state of Missouri alone there were 134 such associations in 1912 with resources of more than 14 million dollars.

As with most financial institutions, so with Building and Loan Associations, there have been grave abuses. But with state supervision there has come safe and competent management. Supervision in Missouri was adopted in 1896. Almost immediately several associations failed. But in the last eleven years there has not been a single failure in the state of Missouri.

The purposes of the Building and Loan Association are: (1) saving, (2) investment, (3) cooperation.

To illustrate the purposes and the principles underlying these associations I draw freely upon the material used by them.

(1) Let us imagine that each of two hundred laborers agree to deposit in a vault $1.00 per month for 200 months. Each person, by this voluntary act, would be able to save this portion of his earnings, and at the end of the stated time would be entitled to $200. This is systematic saving, and savings precede investments.

(2) Let us suppose that the annual deposits at the end of each year are loaned to men outside this group at 8% simple interest. At the end of 200 months each contributor would be entitled to $325.44. Or better still, we may imagine that the deposits of each month are immediately invested at 8%, and compounded monthly. At the end of 200 months each contributor would now be entitled to $421. So, under either supposition here made, less than 200 months would be required to realize $200. In fact

$126\frac{2}{3}$ months would suffice, when the interest is compounded each month at 8%.

(3) It may now happen that some member of this group will desire to buy a building lot for $200. It occurs to him that the first monthly payment of $200 would just meet his needs. He is willing to pay the required interest each month, to give a deed of trust on his lot, and to keep up his monthly deposit of $1.00 for $126\frac{2}{3}$ months until each of the other members will also have $200. At the end of this time, he will neither owe anything to the treasury of this group, nor receive anything therefrom. His dues at 8% interest compounded each month will pay his debt. The interested circle of men see the advantages of the proposition. They will not need to seek a borrower, and the process of investment is simplified, in that there are fewer exchanges of money. The plan is also attractive to the borrower in that it offers unusual advantages for the immediate investment of funds. The time and amount of payments and the compound interest rate, recommend it strongly. But the funds for loans are limited. Hence it is quite possible that several depositors will seek the same loan. Therefore it may readily happen that some will be willing to bid more than the 8% rate required. The bid may be so much per $100 paid monthly. This premium will further increase the earnings of the depositors. So in addition to saving and investing, we now have cooperation in investments and savings that is advantageous to all.

In brief this illustrates the principles underlying Building and Loan Associations.

Under the Corporation Laws of the state of Missouri 25 or more persons may associate themselves together, and having complied with the "Articles of Agreement," be incorporated as a Building and Loan Association. Then with expert management and direction this association may do in a very efficient way the things previously suggested. With honest and efficient management, combined with reasonable rates, such an organization should command the consideration of both those who borrow and those who invest.

The small investor, who becomes a share holder in the association, is offered a ready means for systematically investing his earnings; he is given part in an enterprise where he might not be qualified to assume direct control; he is permitted to participate in great undertakings that are possible only through cooperation and expert direction. The borrower without means, who will mature stock in the association, can obtain funds for building purposes, when it would be very difficult to borrow elsewhere; he is given a chance to apply his savings immediately to the liquidation of his indebtedness; and if his dues are kept paid, he has a long time loan with no expense or worry about renewals. In either case the shareholder has the rare privilege of seeing his money grow month by month after it has left his hand. To the teacher or the salaried worker this is an attractive proposition. Unless we are able to harness our surplus earnings and make them labor for us, the time will soon come when our maximum earning efficiency will be reached; and then in our waning strength we shall be compelled to draw upon our savings which from this point on must of necessity decrease and ultimately face exhaustion. To save money and let it make money is an art that pays. Suppose a child at the age of ten would earn $1.00 a month and place it in his toy bank until he has $6.00, and then deposit it in a savings bank at 8% interest compounded semi-annually. If this saving and semi-annual deposit were continued without interruption for 5 years, the boy would have put $60 into the bank, and would have $74.92 to his credit. In 10 years his deposits would be $120, his credit $185.82; in 15 years his deposits would total $180, his credit $349.70; while in 20 years his deposits would equal $240, and his credit would mount to the magnificent sum of $593.60. Were these deposits made and compounded monthly the credit would be larger still. This is what a well managed Building and Loan Association can do with its monthly receipts.

Assume that the par value of a Building and Loan Association share when paid up is $200, that the monthly payment is $1.00 per share, that the association is earning 12%, and that Henry James buys 10 shares of unpledged stock. The problem is equivalent to this: Mr. James pays to the association $10 per month. The first payment immediately earns 12% interest. At the close of the month his 10 shares are worth $10.10. Add to this $10, the dues paid at the beginning of the second month.

The shares are then worth $20.10. This sum is immediately increased at the designated rate. The shares are worth $20.30 at the close of the second month. Add to this the payment at the beginning of the third month and he has $30.30 invested during this month. Thus month by month through payments made and interest earned the shares increase in value, and when finally by this process each share is worth $200, the stock is said to be matured and Mr. James will receive $2000 and surrender his shares of stock. As we shall see it would take a trifle less than 110 months to mature these shares.

Suppose Cyrus Hill, who desires to build a house, makes application to this same association for a loan of $2000. The established rate of interest on loans is 8%. Mr. Hill bids a premium of $33\frac{1}{3}$ cents per month on each $100. The loan is made. The interest and premium on $2000 is $20 per month in advance. He also pays $10 per month to mature his 10 shares of pledged stock. His monthly dues are therefore $30. Mr. Hill is in reality paying 12% interest on the $2000 borrowed, and receiving 12% interest on his monthly payments of $10 which will also mature his shares in 110 months. He is paying a high rate of interest and in turn receiving a high rate. But up to the time of maturity the principal remains greater then the sum of the payments. Therefore, under the conditions of the problem, Mr. Hill is paying interest in excess of the current rate. Some Building and Loan Associations have fixed premiums, others do not have this feature at all.

In the calculations relative to Mr. James' shares, it may seem that when we find the worth of a share at the end of a given year we need to know its worth the preceding year; and that when we find the time of maturity at a given rate we use a trial process. This leads us to consider the mathematics of Building and Loan Associations.

The compound amount of a dollars at rate r for n years is $a(1+r)^n$. This is true of any sum of money for any period of time where any unit of measure is used, a representing the sum, n the number of these units and r the rate per unit of time. Let us now deposit a dollars annually at the beginning of each year at rate r for n years. Then, if A_1 is the amount of the first deposit at compound interest, A_2 of the second, etc., and S the total amount, we have

$$\begin{aligned} A_1 &= a(1+r)^n \\ A_2 &= a(1+r)^{n-1} \\ &\ \vdots \\ A_n &= a(1+r) \end{aligned}$$

$$\therefore\ S = a[(1+r)+(1+r)^2+\ldots\ldots+(1+r)^n] = \frac{a[(1+r)^{n+1}-(1+r)]}{r}$$

$$\therefore\ S = \frac{a}{r}(1+r)\,[(1+r)^n-1]. \quad (1)$$

This formula, which holds for any unit of time where r is the rate in that unit, is well adapted to logarithmic work. By (1) we can readily find the worth of a Building and Loan share at the end of any given time. For instance if $1.00 is the monthly payment, the rate 8% and the time 200 months we have $a = \$1$, $r = .00\frac{2}{3}$, $n = 200$. Formula (1) gives us amount

$$S = \frac{1}{.0067}(1.0067)\,[(1.0067)^{200}-1].$$

By logarithms $S = \dfrac{(1.0067)\ (2.802)}{.0067} = 421$, the number of dollars a share is worth at the end of 200 months.

From (1) we can also find the time required to mature a share of stock. Take the shares of Mr. James. Here $a = \$1$, $r = .01$, $S = 200$. By (1) $200 = \dfrac{1}{.01}(1.01)\,[(1.01)^n-1]$,

$$(1.01)^n = \frac{(200)\ (.01)+1.01}{1.01} = \frac{3.01}{1.01}.$$

$$\therefore\ n = \frac{\log 3.01 - \log 1.01}{\log 1.01} = 110, \text{ the number of months.}$$

Therefore n = 9 yr. 2 mo. the time it takes Messrs. James and Hill with monthly payments of $1 each to mature stock worth $200 when money is worth 12% compounded monthly.

Because of the high rate of interest involved, (12%), Mr. Hill's shares furnish an interesting problem. On ten shares of stock Mr. Hill pays $10 per month. By formula (1) this amounts to $2000 in 110 months. But the normal rate of interest is about 8%. At this rate by (1) the payments amount to $1,637.60 in 110 months. Therefore he realizes from his dues $362.40 beyond one's expectations. But on the other hand he pays $20 interest each month. This is $6⅔ per month more than would be required at 8%. Were it possible to obtain 8% interest on this excess monthly payment of $6⅔, and compound it monthly, in 110 months it would amount to $1091.73. This is a loss due to excessive interest charge. Now deduct $362.40 gain, and we still have a net loss of $729.33. In this extreme case the borrower pays not only 8% interest, but also $729.33 premium at the end of 110 months; or translated into present worth at 8% compounded monthly, he pays an interest charge of 8% and a premium of $349.90.

In the example just cited the interest charge and the earning rate of the association are each assumed to be 12%. But the earnings may reach a higher rate per cent than the interest. Then the premium above a normal interest rate will decrease or disappear entirely. Had the earning rate in the previous problem been 15% and the interest charge 12%, the results obtained would differ materially. The loan would now mature in 100 months, and Mr. Hill would pay interest equivalent to 8% per annum and an advanced premium of $110 instead of $349.90.

The chief sources of income for the association arise from interest on loans, gains from the purchase and sale of real estate, premiums on loans, fines for non-payment of dues, and profits from withdrawals before the maturity of stock.

The Building and Loan Association is a growing method of cooperative investment. It offers an inviting field for investigation, and suggests many interesting problems to the teacher of mathematics.

In this brief survey we have seen the underlying principles and purposes of the Building and Loan Association, and have examined its claim both upon the investor and borrower.

There is an astonishing imagination, even in the science of mathematics. . . . We repeat, there was far more imagination in the head of Archimedes than in that of Homer.—VOLTAIRE.

AN EXPERIMENT IN TEACHING ALGEBRA*

By Chas. A. Epperson

It is not usual in teaching high school algebra to introduce any of the methods of the analytic geometry to explain the meaning of such equations as those of the straight line, circle and ellipse. As a general thing if these curves are plotted at all it is by locating points on them. The idea of making use of the slope and intercept of a line, the center and radius of a circle, or the center and semi-axes of an ellipse to plot these curves is not introduced.

It was with the end in view of trying to find out whether or not the easier principles of the analytic geometry would be readily understood by a class in the second year high school algebra and of making use of these principles in plotting curves that this experiment was undertaken.

The class was the one called the Third Quarter Algebra Class in the First District Normal School of Missouri. The work indicated for the class was pp. 236-365 of Slaught and Lennis, First Principles of Algebra. The members of the class were for the most part from the rural districts of North Missouri, and had not had an opportunity to do their high school work in high school but were compelled to come to the Normal School for it. The average age of the students was eighteen or nineteen. So much for the class.

By touching very lightly such topics as Highest Common Factor, Lowest Common Multiple and cube root, the class was able to cover the required pages thirteen days in advance of the close of the quarter. In their previous work they had taken up plotting of the straight line, the parabola, the circle, and the ellipse, by means of locating points on them. I then took up in lecture the following topics in the order named; the definition of slope, to find the slope of a given line, the equation of a line in slope-intercept form, the equation of a line joining two points, the equation of a line in intercept form, parallel lines have the same slope, perpendicular lines have slopes which are the negative reciprocal of each other, the length of a line segment, the

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*Reprint from "School Science and Mathematics", December, 1916, by permission of the Editor, Mr. Herbert E. Cobb.

equation of a circle when its center and radius are given, to find the center and radius given the equation of the circle, to write the equation of a tangent to a circle at a point on the circle, to write the equation of an ellipse when its center and semi-axes were given, and to find the center and semi-axes when the equation of the ellipse was given. Proofs were given to every statement except to the one concerning the slopes of two perpendicular lines. Problems illustrating each step were solved at the board and examples set to be handed in the next day. This was the method of the experiment.

On the thirteenth day the following test was set over the twelve days' work.

I. (a) What is meant by the slope of a line?
(b) What is the slope of the line $2x-3y=6$? Of any line which is perpendicular to this line?

II. (a) Find the equation of the line whose slope is 2 and whose y intercept is 6.
(b) Of the line joining the points (3, 4) and (-2, 1).

III. Find the equation of a circle where center is (2, 3) and whose radius is 6.

IV. (a) Find the center and radius of the circle
$$x^2+y^2+2x-6y=16.$$
(b) Find the equation of the tangent to this circle at the point (4, 2).

V. Find the center and semi-axes of the ellipse
$$3x^2+4y^2+12x-12y^2=2.$$

VI. Determine A so that the circle
$$x^2+y^2-4x-6y=A^2-13.$$
shall cut the ellipse
$$\frac{(x-2)^2}{16}+\frac{(y-3)^2}{4}=1$$
in four points.

By considering each question of equal rank and each part of a question of equal rank the average of the twenty papers was 76%. If those who failed in the work be left out the average of the remaining seventeen was 88%.

This is the second experiment of the kind I have conducted. The records of the first were not kept as it was very short, covering only four or five days. I have in mind several such experiments in classes other than this. For example the following problem, "Find the locus of a point whose distances from two given points are in the ration m: n." (Wentworth Smith, Plane Geometry p. 252, problem 11) could be much simplified by introducing the analytic method. The value of this kind of work as I see it is that it gives the student a taste of methods different from any he has known, simplifies many difficult problems, and gives to formerly meaningless equations a very definite meaning.

Since the above account was written another experiment of a similar nature has been conducted. The class was the Fourth Quarter Algebra Class in this school and was the course immediately following the course described above. Conditions were practically the same as regards the character of the students and the amount of work they had done. By omitting some of the material in the text I had nine days at the end of the quarter to devote to the experiment. I took up in lecture, as before, the following topics: the remainder theorem, synthetic division, and Horner's Process for locating the irrational roots of a cubic equation. On the ninth day the following test was given.

I. State and prove the remainder theorem.
II. State the factor theorem and show how it follows from the remainder theorem.
III. Locate to three decimal places one positive root of the equation $x^3+3x^2-2x-1=0$.

The time of the test was unexpectedly shortened to thirty four minutes, so that most of the members of the class did not complete the last problem. However, if we let the last question read "to two decimal places" instead of "to three decimal places" the average grade of the 14 papers was 79%, or if as before we omit two who failed the average of the remaining twelve papers was 84%.

The results of the experiments above recorded seem to indicate that there is a world of material in the college algebra and analytic geometry that might profitably be employed in our high school work. I should like to suggest that teachers of high school mathematics plan and carry out some such experiments as these and thus give to their classes an introduction to what lies beyond.

SOME SUGGESTIONS AND OBSERVATIONS ON THE TEACHING OF HIGH SCHOOL MATHEMATICS

BY G. H. JAMISON

A mathematician of very great ability said only recently to a class that mathematics was not well taught unless it was made difficult. He also suggested that when teachers make the subject easy it is because the teacher does not know the mathematics he is teaching. We have the wrong conception of mathematics when we look upon it as an easy subject. We are poor teachers of the subject when the product of our teaching is some student who says, "I never had any trouble with my high school mathematics." I might make exception here to the very few geniuses in mathematics. A student last summer boasted of the fine grades which she receivd in her high school mathematics, yet she was hardly able to do passable work in one of our easiest college courses in mathematics. She is typical of many students who have high marks in their high school mathematics, but who clearly show that they have no understanding of the subject. It is my judgment that when students boast of their finding mathematics easy that it reflects discredit upon their teacher. Even for the geniuses, the work can be made hard. For the keenest of mathematical minds, there are fields yet unexplored. And in any subject there are problems which will require the hardest work to solve. Mathematics will never be popular until students are keyed up to the desire for the hard task. Whenever in any subject you make students think, then that subject is hard.

It is a serious injustice to a student to allow him to go through a course in mathematics without challenging and drawing out his deepest thinking. A student was askt not long ago how she was progressing in her arithmetic in the Normal School. She replied that she was not doing so well as she had done in the rural school the winter before. She explained that whenever she came upon a difficult problem, or a problem which appeared difficult, her rural school teacher immediately workt it for her. Many students feel that they are doing well and that they have an excellent teacher when the teacher does all the work for them. The truth is that we frequently do most for people when we do nothing for them in the way of actually thinking out the problem for them. Nine times out of ten the pupil will solve the problem which he says he cannot get if only the teacher will encourage him by a few wise directions. It was not an uncommon thing for our parents in the schools of twenty years ago to stay with problems for a week at a time, but now we feel that the student is doing well if he works an hour or so on a problem. My suggestion is that we should seldom work problems for students and when we do, our aim should be not so much to show them how to work that particular problem as to teach them the way they should attack their difficulties and get them for themselves. Let us test our mathematics teaching by this high standard: Have I made pupils think? Do they have power to attack new situations and overcome them? Are they growing in power to grasp mathematical truths?

Whenever a student faces a problem proposed for solution, he should be impressed with the fact that there are two parts to it. One is knowing how to solve the problem; the other, the solving of it. The first thing is to look over the field and see which way to go. The second is to go the way you have planned. The architect knows how the building should be constructed. He plans it. He makes the blueprints of it. The carpenter builds the house. Let me illustrate what I mean by the two-fold nature of every problem. Suppose we are to add several fractions. How do we do it? Answer: We find the lowest common multiple of the denominators. Then we reduce each fraction to an equivalent fraction, having this lowest common multiple for its denominator. Then we add the numerators and divide the result by the common denominator. Most students are able in a mechanical way to do this, but few of them know how they do it.

Ask most any algebra student how he would solve the equation $3x=6$. He cannot tell you how he would solve it, but he can solve it. I find it a most difficult thing to get students who have had a year in algebra to tell how to clear an equation of fractions, but they can do it. They will tell you how they think it is done, but they are usually wrong and the discouraging fact is that they do not realize their error. Who cares what the sum of a thousand algebraic fractions is? From the standpoint of the educational value of problem solving, the most important thing is the way by which it is done and not the result.

It is poor teaching indeed that is satisfied merely with an answer. Answers may be had by the thousands, but the way to do a thing is found by thinking. Instead of saying in mathematics that we learn to do by doing, we had better say we learn what to do by thinking. Frequently when a teacher has gone over a problem with care before a class and then calls for remarks he hears something like this. "It is not right." "What is wrong?" asks the teacher. "I do not know, but it is not the answer in the answer-book," replies the student. That student is looking for a result and he does not care whether the method is right or wrong. How far from the roadway to power is the pupil who reports only this, "I did not get the answer!" I believe it would be a good thing to have errors in an answer book so that students might be taught to be sure of their ground. They ought to be self-reliant. It is heartening to hear a student report that he solved the problem correctly but did not get the result in the answer-book. Until our pupils are more eager to get a correct solution than to get the answer they will not succeed.

Let me suggest that sometimes you have pupils in examinations tell how they do a thing rather than have them do it. In geometry let them outline what is necessary to prove the proposition. Let them in algebra state clearly what they do in solving an equation like this: $13x+4-x+16=32x+6-3x$. Let the work be planned so that the greater emphasis may be placed upon the way rather than the result.

Mathematics is one subject, which, in order to have the best mathematics, requires the best English. It is my judgment that Geometry affords as good a field for the development of good English as any other subject. Mathematics is an exact science, and unless you express its truths with exactness and clearness you have failed. The idea or truth to be exprest must first be clear. It must be in the mind. Then let it be given out so that the thought conforms to the idea. From arithmetic on thruout the high school mathematics, English should be emphasized, not primarily for the sake of the English but for the better mathematics. We frequently in asking questions frame them in such manner that the student has only to nod in assent or use yes or no in reply. That takes from the learner the power gained in a full clear expression of his impressions. When one can give clear expression to anything it strengthens the impression. In fact one does not know a thing so that he may say it as his own until he gives it to others.

Why do students use "would be" so much in such explanations as this: If three-fourths of a number equals twenty-one, then one-fourth of it "would be" one-third of twenty-one or seven. And four-fourths of it "would be" four times seven which is twenty-eight. Ask a student how he reduces three-fourths to twentieths, and usually he replies "Multiply three by five," or he may say, "Multiply the fraction by five." Neither explanation is good English because he does not express the idea he has. Why does a student when he starts to define something begin by saying it is "when" or "where" as in this definition, an angle is where etc., or the L. C. M. is when, etc. The failure to insist on the use of good English is largely the cause of so many failures mentioned elsewhere in the article; namely, the failure to tell how to do a thing that can or already has been done. If the statement of the proof of a theorem ordinarily given in a half dozen sentences can be stated in two or three sentences, it should be done. It will be better training in English and will give a far quicker and better grasp of the mathematics involved. Let me give an example which is typical of a great many. To bisect an angle ABC take B as the center and with any convenient radius describe an arc cutting BC at X. Using the same center and radius, draw an arc cutting BA at Y. Then with X as center and radius XY, draw an arc. With Y as center and same radius, draw another arc intersecting the first in Z. Then connect B and Z by a straight line. This takes sixty-nine words and five sentences. Now notice this solution: To bisect an angle ABC, take B as center and with any convenient radius draw an arc cutting BC and BA at X and Y respectively. With X and Y as centers and radius XY, draw two arcs meeting at Z. Connect B and Z. This solution requires forty-six words and three sentences. In combining the sentences in the first explanation, one connects more closely the related facts. So many exercises and theorems in geometry require the most careful selection of language to give a clear proof that I feel half the value of the subject is lost if we rest content with only an understanding. Let the teacher appoint herself or a pupil as supreme court justice and throw out any proof which is not wholly correct. We need to pay heed to greater accuracy to be able to state a thing clearly and most effectively with never a waste word. So many times one finds students who have the correct idea, but are unable to express it. Then it is time to say, "We will now adjourn the mathematics recitation and make of it a lesson in English." Put upon the student the responsibility of explaining clearly and logically to the end the proof of a theorem. We feel that we can say the thing just exactly right when the student hesitates and so we relieve him of his burden by explaining it ourselves. We do incalculable harm when we teach in that manner. Help the student by letting him fight his problem thru to the end alone.

We need to have more respect for truth in our mathematics teaching. We are utterly devoid of conscience here. We permit pupils to use the most absurd things in explanation. Truth is the chief thing we are after in teaching. If we overlook truth, if we put slight emphasis upon the accuracy of thought to the idea, we are missing the mark in the education of a growing mind. Let me give a typical example illustrating the kind of work that we should be ashamed to tolerate. A man sold a farm for $5000. This is 25% of the cost more than he paid for it. Find the cost. Explanations such as these are usually given:

100% = the cost
25% = the gain
125% = the selling price
125% = $5000
1% = 1-125 of $5000 = $40
100% = 100x$40 = $4000 = the cost

4-4 = the cost
$\frac{1}{4}$ = the gain
5-4 = the selling price
5-4 = $5000
$\frac{1}{4}$ = 1-5 of $5000 = $1000
4-4 = 4x$1000 = $4000 = cost

There is only one thing wrong with both solutions and that is everything is wrong. There is not a true statement in either explanation. Students working this way are mechanically getting an answer. They have not thought else they would not say that 5-4 = $5000. The truth is, if the pupil will look deep enuf to find out, that 5-4 of the cost of the farm in question is $5000. No student can ever get a good grasp of percentage when so much sacrifice of truth is allowed.

I give another illustration which was copied down just as a student left it. 4-5 of the cost of a carriage is $120. What is the cost? Solution: 4-5 of 1-120 = $150. This pupil thought the work was right because she had an answer. Let us ask our students to keep this commandemnt in their mathematics work, "Thou shalt not lie." They are not dishonest thru any intention to be so. Let them be honest thru good intention. This kind of work I have given examples of does not belong alone to the field of elementary arithmetic. It is also the work of students who are graduates of the high school and who are doing work in the Teaching of Arithmetic classes. Challenge every expression in a mathematics recitation by this sword thrust: "Is it true?" If the surgeon's hand is not true, if the engineer's eye sees falsely, if the carpenter's roofing of the house is contrary to the true principles of architecture, then damage results. There must be accuracy and fineness of doing. But before we can have this we must have fineness of thinking, which will end in truth. These lies we tell in our mathematics are not extremely bad lies, but they are bad enuf that a growing youth should be shocked into sensibility when he is apprised of the falsehood. The child's mind must be just as true for its own sake as the surgeon's hand is when it grasps the knife.

We are too much the slaves of textbooks. When a pupil undertakes to explain at the board a problem, he should do it without depending upon a textbook for the facts. Let him get the material in his head and then give it to the class as if it were his own. One of the great causes of failures in our problem solving is that students do not have the facts of the problem in their own minds. They depend upon the eyes to get it from the pages and the eyes fail. Eyes do not have hands to hold facts as heads do.

Another suggestion to teachers of high school mathematics is that they organize mathematics societies. Let the purpose of these clubs be to afford the pupils a chance for investigating some field of mathematics which they have not touched upon in class or which they have studied but little. The history of mathematics affords a field of peculiar and increasing interest for the student of mathematics. Let them study and report on such topics as these which follow: The Life of Pythagoras and The School He Founded, The Early History of Geometry, The Papyrus of Ahmes, The Life of Newton, The Mathematical Activities of the Sixteenth and Seventeen Centuries, and The History of Decimal Fractions. I do not want to suggest that all programs be given wholly to the subject of the history of mathematics; but that is a field which is claiming large attention from the leading writers on mathematics. A history of mathematics such as the one written by W. W. R. Ball or Florian Cajori will furnish material for a year's work. Perhaps such a society should not meet oftener than once in two weeks. I feel that such an effort in this direction would result in much good.

As a result of my observations of students' work in The Teaching of Arithmetic classes I am very strongly of the opinion that high school students need a half year in arithmetic. It is amazing how utterly weak the high school graduates are in that subject. As a general rule, there are but few things they can do well. Arithmetic has so many applications that it seems that the student's final work in mathematics should be a review of this important subject. If he teaches school, he will need it before he can teach arithmetic. If he goes into business or professional life, he needs a good knowledge of arithmetic so that he may handle his own problems. He also needs it in order that he may give aid to his own children. We are not excusable at all if we fail in teaching any branch of mathematics well, but I am sure the public will be less inclined to excuse us if we fail to prepare good students of arithmetic. A good course in arithmetic coming after the pupil has had algebra and geometry will show the light and help which come from those quarters.

Let me suggest as a final word that we take with us into our classrooms the thing which inspired Poincare, one of the greatest mathematicians of recent times, and which he exprest in these words: "The search for truth must be the goal of our activity; it is the only end that would be worthy of it."

THE PURPOSE AND CONTENT OF HIGH SCHOOL ARITHMETIC

By Byron Cosby

What we are to do for the child will in a large measure determine the purpose and content of any subject. If we are to develop the personality of the child, taking the material of the world that the child has not yet experienced and making it over into the life and experiences of the child, we will choose different material than if we are to make a machine of him for tabulating and collecting data. I believe that the child should be considered a rational being with a chance to grow, to laugh, to play, to work, to think, to construct, to express and anything else that will make life a bigger field of useful experiences, both individual and social. People are different. Personality is to make for distinctness among persons.

President William DeWitt Hyde, of Bowdoin College, in a delightful essay entitled The Teacher's Philosophy, states that the development of personality calls for five well defined and distinct steps. These steps are listed as the different interests that one passes thru in his growth from childish things to that age where one is supposed to put aside childish things.

The Primary or first interests are found in all children, and even in the elementary grades, and are the single suggested interests. The play instinct is dominant. I visited the Kindergarten and noticed that the children were all playing but not at the same game and not any of them intensely for a very long period of time with the same interest. I noticed that the same state prevailed in other grades and even in the life outside of school. These interests are found among all men. We find it represented in baseball, football, the theatre, the fraternal lodge, and even business. This stage calls for many and varied interests throughout the day. We do not expect very long or studious effort to be put upon any question. Those who never leave this stage make life easy in a manner, because they can leave a situation as soon as the effort is a trifle heavy. School teachers change positions frequently.

Following the first interests are those that we stick to on account of results that are more intimately connected with certain results or possibilities. In school the boy looks after the advantages or the penalties. The same continues throughout the active an restful life of many men. The advantages of position, or of mone or of anything else makes them stick to a problem long after th problem has lost all interest. We leave out interests often because we cannot find any opportunity for using them in our chosen craft. These interests are not entirely confined to the upper elementary grades but perhaps belong to that age more than to any other.

The High School interests are often individual. The ordinary high school student thinks of his problem. He has not yet oriented himself with respect to society. Some say he is selfish. He fails to discriminate and differentiate, except as the question relates to his individual life. Here the school work must be individual not wholesale. You cannot teach the class, but must reveal to the student himself, the thing that he has been looking for and has not yet discovered and convince him that he has found his quest. To do this the teacher must have a purpose of his own; must have something in his developed character that is not found in every individual that the child comes in contact with. When a group of high school boys and girls are restless and ill at ease, with low deportment, and careless in discipline there is only one thing that is decidedly wrong and that is the teacher has not lived long enough or else has lived too narrowly and inefficiently. In order to teach in a high school one must have not only experienced his own life, but must have experienced the lives of many by association, by reading, by traveling, by acting, by playing, by singing or any other and all other activities that the social world approves as legitimate. He has to have done this for his work is a work of discovering many and varied promptings and ideals.

To carry the problem through Dr. Hyde states that the college interests are social and the University interests center around thought. The University teaches to respect thought, the college and Normal School to love the world and the human beings who make up the world and its activities. The college and Normal school help you to be agreeable, the University to be a thinker.

With these interests before us as a basis for argument let us examine our problem.

The purpose of the high school arithmetic class, along with other high school subjects is to help the boy or girl to find himself or herself. It is to help the backward boy to throw off the early or primary interests, to leave forever the weighted interests and find himself in the social organization. But one cannot teach others how to fit into the social organization if he has not learned the rules of the game, nor who has not given himself a chance to know the rules of the game so well that he is able to take away as well as to give. The high school boy or girl, with the spirit of selfishness, the lack of discrimination, the inability to differentiate, must have his problems so grouped that he sees some relation between the problem and his life and ambitions. He is not looking entirely to the future, but more nearly to the present. Not many boys have decided upon a craft. The best way to help the high school boy or girl to put aside the childish things and fit into the social groups of the college and normal school and to respect truth, knowing that the truth will give freedom, is to help him in terms that he understands and in the units of measure that he wants to work with. The high school student measures his magnitudes in new dresses, new ties or other articles of wear, in automobiles, in parties and in dollars. Our task is, then to help him organize his interests in terms of these problems and the problems that he associates with his interests, and then advance to the ideal by relating these problems to the real problems in life. The aim is to help the student to become financially, socially, morally, physically and ethically efficient, in order that he may be helpful to himself and others and to be happy when alone and when with others.

The student is to be taught that the life in the school room and outside are the same, and that the whole is a democracy, where no one can do the total thinking for all, and every one must contribute his part to the communal welfare. He must realize that he must look to the whole for part of his information and knowledge. Our quest for practicability must not lose sight of an inherent desire of conquest. The child has a right to demand that he have a chance to think.

The content of the high school arithmetic is a reproduction of those experiences that the race has found valuable or is to find valuable in the near future. It not only must reproduce those experiences but must reproduce them in their developed setting and in terms that one can understand and appreciate. The experiences that are worth while ordinarily happen or occur as a unity, or a continuity, and have a decided place in the world's arrangement. Again experiences happen in large blocks and not in so many pages or in so many exercises. A problem in the world's activities happens through long periods of time and are so complicated that many activities are touched, and many situations developed. This means that the teacher of high school arithmetic must have experienced human activities in the way that they happen in the world and not as bits of hash, memorized from some book without the human or social setting. It means that the high school teacher of arithmetic must have recognized the various problems in the present day world and have ingenuity enough to organize, and patience enough to work until the organization is effected.

We discovered several years ago that the proper way to study the classics in French and English was to take the complete drama or story and study it as a whole. Of course there are still some who think that to teach an English classic means to dissect it and label every nerve and sinew with some name that any human being with respect for himself does not care to remember. But in spite of the fact that most of the teaching of the classics has been taught as a completed whole we have been slow to teach our arithmetic as a continuous subject and still find in our newest texts on arithmetic, that the author has dissected and labelled every problem, not paying any attention to whether or not the experience has a place in the world and even tho it has a place whether or not it is useful, beautiful or interesting.

In order to get a high school boy or girl to understand the problems in his language, and in terms of his own selfish undifferentiated interests I think one might show him the economics of education, how to build a house, operate an automobile or any other of the reasonable experiences that the average boy or girl goes through almost any day in the year. The young man wants to know how he can get rich, how he can save energy, how he can receive rewards without much effort, how he can avoid penalties, how he can understand the machines that are around him, how he can make his play or recreation bring him greater happiness, how he can please his friends, how he can operate and discover the working principles of all forces, and anything else that happens in his vicinity, or that he hears of or reads about. He wants to do on the tomorrow what his heroes are doing today, only he wants to do it with greater ease and cleverness. He wants to improve. His ambitions and ideals are high.

The problem is more attractive when the child sees the need or the relation. A series of problems, not exercises, showing the value of an education from the arithmetic side as a factor in the making of the living, will help the student to see the need of studying those things that will make life richer and fuller at the present and in the future. The problem should be one that gives the student the same experience in the working that he will need in attacking a problem in his business or play in after life. An economic problem stated in arithmetical notation makes clear the fact more readily than any other notation. The waste of labor, capital, material, thought, taste and other human elements can be measured in dollars. If idleness can be measured in dollars, and waste of material shown to be a depreciation, in capital, the student has experienced a problem of evaluating labor and capital.

Does it pay to get a high school education from a financial standpoint? Does increased financial income up to a certain point make a man more efficient, more useful and more happy? Are you of greater service to yourself and to society if your earning power is increased? If you are not able to earn enough to support yourself, provide for the needs of your family, promote the general or communal welfare, build up a library or other working tools, travel, and have the other cultural and useful conveniences can you be of the highest use to society or make your life the most attractive to yourself, your family, and your country? A man with a common school education is supposed to be worth in earning power expressed in dollars one and a half times as much as the illiterate man, that is he earns one and a half times as much; while the high school graduate earns twice as much and the college graduate four times as much. And there are other things as appreciation, pleasure, culmination of ideals, and ambitions.

Statistics taken from the United States Census may help us to see the need of trying to increase our earning power. Or someone states that sixty-six out of every one hundred persons dying in this country die without an estate. Only two per cent may be classed as earning sufficient to be socially efficient. Ninety eight per cent are living on the daily wage, are supported by relatives or are living on inherited property. Ninety seven per cent of all people reaching the age of sixty five are dependent upon others for support. With the economic as a basis to start with, the ideal of unity, the characteristic curiosity, conquest, the assembling or synthesizing, and analysis let us organize a problem.

Instead of choosing several so called problems, but in truth exercises, about lumber measure, cement, plastering, wallpapering, painting and similar problems, take the building of a house. It involves all of the tastes and instincts of the class no matter how varied the membership or the previous experiences of the class. One student is interested in drawing the plans, another in excavating and removing the dirt, another in the mixing of the cement, and another the laying of joists and the building of walls. While some of the students have found themselves in this kind of work others are working on the amount of heavy lumber needed, others on the various types and quantities of lumber to be used, others on the millwork, others hardware, others heating plants, others systems of lighting, and others on papering, painting and interior decoration. Each solves the entire problem, but only as he is a member of the class and only in the way that he contributes to the whole. Each takes for his work the part that he is interested in, and has terms or units of measure for expressing himself, and has had experiences in, either personal or through reading or listening to others. I found in trying such a problem in our Practice School that every boy could be interested in his part. A part of the class took up the organization of the forces necessary to build the house looking up the abstract of title, the deed, the deed of trust or mortgage, the check for the paying and the promissory note accompanying the deed of trust and gave his information to a class that was interested in the whole problem, and needed this particular student's work to help to unify the whole. Each received only that for which through experience and work he was ready for. The girl with a liking for stenciling and designing worked eagerly on her stencil borders for the dining room, the living room, the parlor, and the library, drawing her illustrations to scale and doing exactly the same problem that the boy did in determining the amount of lumber needed. She had to measure and do it with as fine a degree of nicety, as the boy who wanted to know the number of studding. The reports that each member of the class made to the whole were attractive, because the one reporting put his life, his interests, his energy, and his investment into the problem.

As another type of a problem meeting the manufacturing situations take the automobile industry. Here we need to assemble the problems of expense, for labor, capital, depreciation, material, upkeep, interest on investment, replacement value, as new machinery is invented and dozens of other items that must necessarily enter into the operating expenses of a manufacturing business. The student also has to find out the cost of marketing, the cost of delivering to station or shipping depot, the railway charges, the cost of billing out, the cost of following up in case of failure to deliver, the retaining of the trade, the cost of salesmen, the overhead charges, the publicity or advertising expenses, the profit and everything else that might enter into the business. Most of our failures in business, and Dun and Bradstreet's agencies report a very large percentage, is due to the fact that business men rarely know the actual cost of anything. They may know the initial cost, but the advertising cost, the overhead charges, the drayage and other items are lost track of and consequently many things are sold at a price actually below the cost of buying and selling. These are unit problems and have a real value and should be found in the High School arithmetics. Such a problem might take many weeks to solve but when solved has actual value. In any and every neighborhood such problems abound. In the country is found the farmer, the small manufacturer, as carpenter, blacksmith, and molasses maker, and in the nearby towns almost every possible sort of business. In our town we find the grocery store, the hardware store, the bank, the light plant, the ice plant, the shoe factory, the pickle factory, the packing plant, the railway offices, the freight depots and their peculiar business, the newspaper, the Western Union offices, the drug stores, the dry good stores, the men furnishings store, the blacksmith shop, the millinery store, the broom factory, the iron foundry, the bottling works, the gas plant, the shoe repairing plant, the law offices, the oil industry, the coal mines and their offices, the farm, the hospitals, the household expenses, the carpenter and contractors, the motion picture business, the paving contractors, the interior decorators, the insurance business and any number of others.

As a special or type lesson on one of these problems we took the furnishing and decorating of an ideal dining room. In this problem a floor plan was drawn to scale, and height of ceiling decided on, the plan of the floor, whether bordered with oak, or carpeted. The painting and finishing called for the relative cost of paint, kalsomine, paper, varnish and floor wax. The question of having the floor border of oak, of matting or of plain varnished yellow pine came up for discussion. Then came the furnishing of the room. The number and quality of material needed, as the number of chairs, the number of pieces of table linen, the number of the different pieces of silverware, the amount of other things found in the dining room, the pictures on the wall, and a general plan of marking the table linen, as with letter or monogram. After the goods were decided upon the list and approximate price determined upon, the local merchants were asked to make bids, and likewise merchants in other towns. If ordering from other towns the question of postage, freight, parcel post or express rates came up, as well as the cost of the money order or draft. The letter was written for quotations, the draft written out and the order sent in the regular accepted manner. This problem gave each girl the chance to make her problem one of her own. She put into the problem whatever she wanted to and yet had to at all times make or secure from some source her data, and then organize the data into terms that she and the other members of the class could appreciate. It is needless to say that each understood the problem and worked with an aggressiveness that cannot be equalled by the old plan of finding the least common mutiple, or spending several hours on simple interest problems, or working for months on denominate numbers.

Another interesting problem was on the cost of an elementary education. It was surprising how much the cost was and how eagerly the children looked for material and brought to light the most unexpected material and decided that such material belonged to the problem and had a part in its solution.

The High School student is inexperienced in synthesis; he organizes with difficulty and not having had the experience of trying to put into words his activities and experiences interprets his problem slowly and with difficulty. He needs experience in organizing, in synthesis and he gets this experience only in practice. He can get this experience or practice in a natural wholesome manner only by collecting his material and working it over into the problem. We understand the things that are written in terms of our experiences, and in our language, even tho the grammar of that language may not always be correct. The child's problem must be stated in terms of his experience and knowledge. The trouble at the present time and with almost every text is that the exercises are ones that the child and oftentimes the adult has never met and are stated in terms that no one understands, or if understood do not interest because so far removed from life or present day activities. Our conclusion is then that the boy and girl must make the content of arithmetic in the terms of the child's experience, in language that the child understands, in the large units of actual every day happenings, far removed from isolated or strained situations, and in present time. The child wishes to be 100 per cent efficient in some one thing and to know many things well or partially.

If arithmetic is to be retained in our high schools it must be retained on account of its simplicity and usefulness. Its simplicity will be reached only when the notation is such that the ordinary person can understand the same without having previously taken a course in physics or the other high school subjects, and it will be useful only when we repeat in the school room those experiences and those only which the race has found helpful or expects to immediately make useful on the tomorrow; these experiences are those found in the larger blocks or units of human activity. No isolated, formal, predigested, strained, dried and dwarfed exercises have a place in the school room. Arithmetic is not a dry subject. Arithmetic not only must be simple and give to us an understanding of those experiences that the race has or will find useful but it should be such a subject as to give to the child the next step in his human experiences. The school room is part of the life of the child and as such it should give human experiences that are valuable and at all times give the same in the most pleasant and economical manner as possible. We need those experiences that are useful, agreeable and that are liable to occur in one's life.

As a resume I think I might say that the result is:

The child works with a known purpose. There is no guess work. He is familiar with the setting of the problem, because he has put it into words himself. He is familiar because he has made the problem only as it has grown out of his experiences and his next step in development, and being based upon his experiences, and on data that he collected, it is worked with an interest that the ordinary book problem can never inspire.

It trains in organizing ability. Facts must be organized in sequence. It gives confidence in the subject and self respect to the student.

It develops originality and independence. "It helps to weigh evidence, draw inferences, make comparisons, invent solutions and form judgments." (Elliott)

Brings home, community and school into closer relation.

Gives live and vital work, as practical utility is a good basis for arithmetical study.

The method follows the law of natural growth, the problem is graded to the needs of the child. He uses just as much of the problem as his development calls for.

It gives a motive.

The child finds truth by his own effort.

The child works with his own tools, and we all work better with our own tools than with some one else's.

It is the only human way. It is the only way that gives the child the next step in his experience, and gives it in the original setting and in a setting that he will meet the problem in in after life.

It fulfills the demand for conquest.

It meets the demand of every child, that he be given a chance to think.

THE CUBE ROOT OF A BINOMIAL SURD

By Wm. H. Zeigel

We are familiar with the method of extracting the square root of a binomial surd, but the extraction of the cube root is not so easy. For four years or more, I have used a METHOD of INSPECTION that leads easily to the correct solutions of such forms when the results involve only rational numbers and quadratic surds. In the solution of the cubic by Cardan's method, there arises the necessity for taking the cube root of a surd of the form $a+s\sqrt{b}$. In Hall and Knight's Higher Algebra, page 70, a method is given which the Authors recognize depends upon the use of the factor theorem in the solution of the cubic. Arthur C. Johnson gives practically the same method in the December number of The American Mathematical Monthly, 1913. In solving $x^3+x-2=0$ by Cardan's method the cube root of $27+6\sqrt{21}$ is required. This leads to the cubic $4x^3+9x-27=0$, which is more complicated than the original equation and more difficult to solve by inspection. And if, to avoid the method of inspection, the writer had also applied Cardan's solution to $4x^3+9x-27=0$, he would have been obliged to take the cube root of $27+6\sqrt{21}$ which was the original problem.

Cardan's method reduces all cubics to the form $x^3+px+q=0$, and finally gets

$$x=\sqrt[3]{D}+\sqrt[3]{E};\ x=w\sqrt[3]{D}+w^2\sqrt[3]{E};\ x=w^2\sqrt[3]{D}+w\sqrt[3]{E}.$$

Where $\sqrt[3]{D}=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$; $\sqrt[3]{E}=\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$, and w and w^2 are the imaginary cube roots of unity. (See any College Algebra). When $\frac{q^2}{4}+\frac{p^3}{27}<0$ the roots of the cubic are all real and Horner's method is generally used. Nevertheless as we shall see our method may apply here also. When $\frac{q^2}{4}+\frac{p^3}{27}>0$ there is only one real root, and in case the factor theorem is not used, we have a real need for extracting the cube root of a surd of the form $a+s\sqrt{b}$ or $a-s\sqrt{b}$, $\sqrt{b}$ in its simplest form, and b positive.

It is evident by using, when necessary, a fractional co-efficient for the whole surd that a and s can always be made positive integers. Furthermore we do not need to assume $\sqrt[3]{a+s\sqrt{b}}=m\sqrt{x}+n\sqrt{y}$ for on cubing, every term on the right hand side would be irrational and we cannot equate rational and irrational parts.

Therefore we may assume

(1) $\sqrt[3]{a+s\sqrt{b}}=x+n\sqrt{y}$, x, y and n positive and $\sqrt{y}$ in its simplest form.

Cubing (1), we have

$a+s\sqrt{b}=x^3+3x^2n\sqrt{y}+3xn^2y+n^3y\sqrt{y}$.

Therefore

(2) $y=b$.

(3) $x^3+3xn^2y=a$, or $x^3+3xn^2b=a$.

(4) $3x^2n+n^3y=s$, or $3x^2n+n^3b=s$. Also

$a-s\sqrt{b}=x^3-3x^2n\sqrt{y}+3xn^2y-n^3y\sqrt{y}$.

Therefore

(5) $\sqrt[3]{a-s\sqrt{b}}=x-n\sqrt{y}$. Hence from (1), (2) and (5) we may write

(6) $\sqrt[3]{a+s\sqrt{b}}=x+n\sqrt{b}$, and

(7) $\sqrt[3]{a-s\sqrt{b}}=x-n\sqrt{b}$.

From (6) and (7) we have

(8) $\sqrt[3]{a^2-bs^2}=x^2-bn^2=I$, an integer provided (1) is true.

Or we may write

(8) $bn^2=x^2-I$.

Substituting the value of bn^2 from (8) in (3) we obtain

(9) $4x^3-3Ix=a$, or $x(4x^2-3I)=a$.

From (8) and (4), we have

(10) $4bn^3+3In=s$, or $n(4bn^2+3I)=s$.

Now suppose $x=\frac{g}{h}$, a fraction in its lowest terms. Then (9) becomes

(11) $\frac{g}{h}\left(\frac{4g^2-3Ih^2}{h^2}\right)=a$.

(a) If h is odd $4g^2$ and h^2 are prime to each other. Therefore $4g^2-3Ih^2$, and h^2 are prime to each other. Hence $\frac{g}{h}(\frac{4g^2-3Ih^2}{h^2})$ is a fraction in its lowest terms and cannot equal the integer a. Therefore x cannot equal a fraction whose denominator is odd.

(b) If in (9) $x=\frac{g}{h}$, $h=2L$, L not equal to 1, we have

(12) $\frac{g}{2L}(\frac{g^2-3IL^2}{L^2})=a$.

As above $\frac{g}{2L}(\frac{g^2-3IL^2}{L^2})$ is a fraction either in its lowest terms, or at most numerator and denominator contain the common factor 2, and therefore cannot equal a. Hence $x=\frac{g}{h}$ cannot be a fraction with an even denominator larger than 2.

(c) Suppose $x=\frac{g}{2}$, then (9) becomes

(13) $\frac{g}{2}(g^2-3I)=a$, g odd. Then if I is also odd, which is possible from (8), $\frac{g}{2}(g^2-3I)$ may equal the integer a as far as (9) is concerned.

Now suppose $n=\frac{e}{f}$, a fraction in its lowest terms. From (10) we have (14) $\frac{e}{f}(\frac{4be^2+3If^2}{f^2})=s$.

(d) Take f odd. Since b cannot contain a factor that is a perfect square, it is at once evident that the first member of (14) is a fraction and cannot equal the integer s. Hence $n=\frac{e}{f}$ is impossible when f is odd.

(e) Let $n=\frac{e}{f}$, $f=2L$, L not equal to 1. Then (14) becomes

$\frac{e}{2L}\left(\frac{be^2+3IL^2}{L^2}\right)=s$. The first member of this equation is a fraction, even though $b=f$. Hence it cannot equal the integer s when f is an even number greater than 2.

(f) If $f=2$, (14) gives

(15) $\frac{e}{2}(be^2+3I)=s$. When b and I are both even or both odd, the first member of (15) may equal the integer s as far as (10) is concerned. It therefore follows that if x and n are fractions they must each have 2 for a denominator.

If in (3) we put $x=\frac{g}{2}$ it gives us $n^2=\frac{8a-g^3}{12gb}=\frac{\text{odd number}}{\text{even number}}$. Therefore n is a fraction when x is. But since the only fractional form each can assume has 2 for a denominator, when $x=\frac{g}{2}$, $n=\frac{e}{2}$.

Conversely we find when $n=\frac{e}{2}$, $x=\frac{g}{2}$. For (4) gives us

$x^2=\frac{8s-e^3b}{12e}=\frac{\text{odd number}}{\text{even number}}$, when b is odd. If b is even it must be of the form $b=2k$ where k is odd. Therefore

$$x^2=\frac{8s-2ke^3}{12e}=\frac{4s-ke^3}{6e}=\frac{\text{odd number}}{\text{even number}}.$$

Consequently x is a fraction when n is a fraction as stated above. Hence for x and n we need to consider only positive integral values, and positive fractions where $x=\frac{g}{2}$, $n=\frac{e}{2}$ simultaneously.

Formula (8) furnishes us a ready means for determining x and n while (3) and (4) readily show us the upper and lower range of values for x and n, together with the final check on the correct solution.

(A) Suppose our problem is to find the cube root of $1656-752\sqrt{5}$. $a=1656$, $s=752$, $b=5$. If the result can be expressed in the form of a quadratic surd we have from (8)

$\sqrt[3]{(1656)^2-(752)^2(5)}=x^2-5n^2$, or
$x^2-5n^2=-44$. Therefore

$$(8)^1 \quad n^2=\frac{x^2+44}{5}.$$

$$\left.\begin{array}{l}x=1\\ n=3\end{array}\right\} \qquad \left.\begin{array}{l}x=6\\ n=4\end{array}\right\} \qquad \left.\begin{array}{l}x=9\\ n=5\end{array}\right\} \text{and etc.}$$

For this problem (3) and (4) become

(16) $x^3+15xn^2=1656$, and

(17) $3x^2n+5n^3=752$ respectively.

It is evident that $x=1$, $n=3$ gives a result that is too small to be a solution of (16) and (17), while $x=9$, $n=5$ and all succeeding solutions of $(8)^1$ give a result that is too large. Hence the solution $x=6$, $n=4$ is tried and is found to satisfy both (16) and (17). Therefore

$$\sqrt[3]{1656-752\sqrt{5}}=6-4\sqrt{5}, \text{ and } \sqrt[3]{1656+752\sqrt{5}}=6+4\sqrt{5}.$$

(B) The application of these principles in the solution of the cubic $z^3-11z+20=0$ will be of interest. As previously stated $z=\sqrt[3]{D}+\sqrt[3]{E}$; $z=w\sqrt[3]{D}+w^2\sqrt[3]{E}$; $z=w^2\sqrt[3]{D}+w\sqrt[3]{E}$; where
$w=\dfrac{-1+\sqrt{-3}}{2}$, $w^2=\dfrac{-1-\sqrt{-3}}{2}$.

In our equation $p=-11$, $q=20$. Therefore

$$\sqrt[3]{D}=\sqrt[3]{-10+\sqrt{\frac{(20)^2}{4}+\frac{(-11)^3}{27}}}\text{ , or}$$

$$\sqrt[3]{D}=-\tfrac{1}{3}\sqrt[3]{270-111\sqrt{3}}.$$

By (8) $\sqrt[3]{(270)^2-(111)^2(3)}=x^2-3n^2$. Hence $x^2-3n^2=33$, or

$$n^2=\frac{x^2-33}{3}.$$

$$\left.\begin{array}{l}x=6\\ n=1\end{array}\right\} \qquad \left.\begin{array}{l}x=9\\ n=4\end{array}\right\} \text{etc.}$$

Equations (3) and (4) become

(18) $x^3+9xn^2=270$, and

(19) $3x^2n+3n^3=111$ respectively.

$x=6$, $n=1$ satisfies both (18) and (19).

Therefore

$\sqrt[3]{D}=-\frac{1}{3}(6-\sqrt{3})$.

$\sqrt[3]{E}=-\frac{1}{3}(6+\sqrt{3})$. Hence

$z=\sqrt[3]{D}+\sqrt[3]{E}=-4$; $z=w\sqrt[3]{D}+w^2\sqrt[3]{E}=2+\sqrt{-1}$;

$z=w^2\sqrt[3]{D}+w\sqrt[3]{E}=2-\sqrt{-1}$.

(C) Suppose we now consider the cubic $x^3+x-2=0$, mentioned in The American Mathematical Monthly, whose solution by Cardan's formula requires the cube root of $27+6\sqrt{21}$. Formula (8) gives us $\sqrt[3]{(27)^2-(6)^2(21)}=x^2-21n^2$, or

$x^2-21n^2=-3$, or

(20) $x^2=21n^2-3$.

Formulae (3) and (4) become

(21) $x^3+63xn^2=27$, and

(22) $3x^2n+21n^3=6$. From (21) and (22) it is evident that any integral values whatever will make the first members of (21) and (22) too large. We shall then need to take the forms $x=\frac{g}{2}$ and $n=\frac{e}{2}$. $n=\frac{1}{2}$, $x=\frac{3}{2}$ satisfies both (21) and (22). Hence

$$\sqrt[3]{27+6\sqrt{21}}=\frac{3}{2}+\frac{1}{2}\sqrt{21}.$$

This method of inspection may be extended and holds when the required root may be found in the form $m\sqrt{x}+n\sqrt{y}$.

Reasoning as before the given surd will take the form $a\sqrt{c}+s\sqrt{b}$, a, s, c, b positive. Accordingly we have

(23) $\sqrt[3]{a\sqrt{c}+s\sqrt{b}}=m\sqrt{x}+n\sqrt{y}$, m and n positive.

(24) $\sqrt[3]{a\sqrt{c}-s\sqrt{b}}=m\sqrt{x}-n\sqrt{y}$.

(25) $x=c$, $y=b$.

(26) $\sqrt[3]{a^2c-s^2b}=m^2c-n^2b=I$, an integer.

(27) $m^3c+3mn^2b=a$.

(28) $3m^2nc+n^3b=s$.

By an argument identical with that used in (a) (b), (c), (d), (e) and (f) we can show that m and n are both positive whole numbers or both positive fractions where $m=\frac{g}{2}$, $n=\frac{e}{2}$.

(D) Find the cube root of $2261\sqrt{2}-1845\sqrt{3}$
$a=2261$, $c=2$, $s=1845$, $b=3$.
By (26)

$$\sqrt[3]{(2261)^2(2)-(1845)^2(3)}=m^2c-n^2b.$$

Therefore

$$m^2c-n^2b=23, \text{ or}$$
$$2m^2-3n^2=23.$$

$$(29)\quad m^2=\frac{23+3n^2}{2}.$$

$$\left.\begin{array}{l} n=5 \\ m=7 \end{array}\right\} \text{etc.}$$

Formulae (27) and (28) give us in order

$$(30)\quad 2m^3+9mn^2=2261, \text{ and}$$
$$(31)\quad 6m^2n+3n^3=1845.$$

The solution $n=5$, $m=7$ of (29) satisfies both (30) and (31). Hence $\sqrt[3]{2261\sqrt{2}-1845\sqrt{3}}=7\sqrt{2}-5\sqrt{3}$, and

$$\sqrt[3]{2261\sqrt{2}+1845\sqrt{3}}=7\sqrt{2}+5\sqrt{3}.$$

If now we permit either b or c to become negative, our formulae with slight changes still hold. Suppose b and c are both negative. In formulae (23) to (28) inclusive, where a and s are positive, m and n will evidently be negative.

(E) With this in mind, we can find the cube root of
$418\sqrt{-5}+369\sqrt{-7}$.
$a=418$, $c=-5$, $s=369$, $b=-7$.
By (26) $\sqrt[3]{(418)^2(-5)-(369)^2(-7)}=m^2c-n^2b$, or
$7n^2-5m^2=43$.

$$(32)\quad n^2=\frac{5m^2+43}{7}. \qquad \left.\begin{array}{l} m=-2 \\ n=-3 \end{array}\right\} \text{etc.}$$

Formulae (27) and (28) become

$$(33)\quad -5m^3-21mn^2=418, \text{ and}$$
$$(34)\quad -15m^2n-7n^3=369.$$

$m=-2$, $n=-3$, the solution of (32) satisfies both (33) and (34). Therefore $\sqrt[3]{418\sqrt{-5}+369\sqrt{-7}}=-2\sqrt{-5}-3\sqrt{-7}$, and $\sqrt[3]{418\sqrt{-5}-369\sqrt{-7}}=-2\sqrt{-5}+3\sqrt{-7}$.

Indeed when b and c are negative in $a\sqrt{c}+s\sqrt{b}$ we only need consider the forms where a and s are positive. For as we have seen a can always be made positive; and were s negative we could take the conjugate of the given number. In this case m and n are always negative. When the cube root is thus obtained, write it again with the sign of n changed. This expression is the cube root of the required number.

By making corresponding changes in formulae (1) to (10) inclusive, they may be used to extract the cube root of the complex number $a+s\sqrt{-\beta}$ or its conjugate, provided the results involve only rational numbers and quadratic surds. In cubic equations where $\frac{q^2}{4}+\frac{p^3}{27}<0$, all the roots are real. If these roots can also be expressed in terms of rational numbers and quadratic surds the above method applied to such problems as $y^3-15y+4=0$ (Problem 1, page 35, Dickson's Elementary Theory of Equations) has some advantages over the method in art. 5 page 35, or art. 7 page 37 of the previously named text.

What is physical is subject to the laws of mathematics, and what is spiritual to the laws of God, and the laws of mathematics are but the expression of the thoughts of God.—THOMAS HILL.

The idea that aptitude for mathematics is rarer than aptitude for other subjects is merely an illusion which is caused by belated or neglected beginners.—J. F. HERBART.

The moving power of mathematical invention is not reasoning but imagination.—A. DE MORGAN.

Analysis and natural philosophy owe their most important discoveries to this fruitful means, which is called induction. Newton was indebted for his theorem of the binomial and the principle of Universal gravity.—LAPLACE.

Mathematics, like dialectics, is an organ of the inner higher sense; in its execution it is an art like eloquence. Both alike care nothing for content, to both nothing is of value but the form. It is immaterial to mathematics whether it computes pennies or guineas, to rhetoric whether it defends truth or error.—GOETHE.